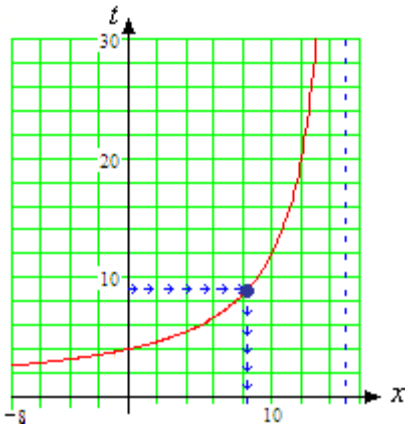
# Lesson 39 Equations with Algebraic Fractions

Francine plans a 60-mile training run on her cycle-plane. The time required for e training run, in terms of the wind speed, $x$, is given by

$$t = f(x) = \frac{60}{15 - x}$$

If it takes Francine 9 hours to cover 60 miles, what is the speed of the wind? We can answer this question by reading values from the graph of $f$, as shown at right. When $t = 9$, the value of $x$ is between $8$ and $9$, so the wind speed is between 8 and 9 miles per hour.




## Solving Algebraically

If we need a more accurate value for the wind speed, we can solve the equation

$$\frac{60}{15 - x} = 9$$

To start, we multiply each side of the equation by the denominator of the fraction. This will clear the fraction and give us an equivalent equation without fractions.

**Example 1** Solve the equation $\dfrac{60}{15 - x} = 9$

**Solution** We multiply both sides of the equation by $(15 - x)$ to obtain

$$(15 - x)\frac{60}{15 - x} = 9(15 - x)$$
$$60 = 9(15 - x) \qquad \text{Apply the distributive law.}$$

From here we proceed as usual.

$$60 = 135 - 9x \qquad \text{Subtract } 135 \text{ from both sides.}$$
$$-75 = -9x \qquad \text{Divide by } -9.$$
$$8.\overline{3} = x$$

The wind speed was $8.\overline{3}$ or $8\frac{1}{3}$ miles per hour.

**Exercise 1** Solve $\dfrac{x^2}{x + 4} = 2$

## Using an LCD to Clear Fractions

If the equation contains more than one fraction, we can clear all the denominators at once by multiplying both sides by the LCD of the fractions.

**Example 2** Solve $\dfrac{3}{4} = 8 - \dfrac{2x + 11}{x - 5}$

**Solution** The LCD for the two fractions in the equation is $4(x-5)$. We multiply both sides of the equation by $4(x-5)$.

$$4(x-5)\left(\frac{3}{4}\right) = \left(8 - \frac{2x+11}{x-5}\right)\cdot 4(x-5) \qquad \text{Apply the distributive law.}$$

$$4(x-5)\left(\frac{3}{4}\right) = 4(x-5)(8) - 4(x-5)\left(\frac{2x+11}{x-5}\right)$$

$$3(x-5) = 32(x-5) - 4(2x+11)$$

We proceed as usual to complete the solution. First we use the distributive law to remove parentheses.

$$3x - 15 = 32x - 160 - 8x - 44 \qquad \text{Combine like terms.}$$

$$3x - 15 = 24x - 204$$

$$-21x = -189$$

$$x = 9$$

**Caution!** We must multiply each term of an equation by the LCD, whether or not the term is a fraction. In Example 2, we multiply each term of the equation by the LCD, including the 8.

**Exercise 2** Solve $\dfrac{1}{x-2} + \dfrac{2}{x} = 1$

## Proportions

A **proportion** is a statement that two ratios are equal. For example,

$$\frac{7}{5} = \frac{x}{6}$$

To solve this proportion, we multiply both sides by the LCD, 30, to get

$$30\left(\frac{7}{5}\right) = \left(\frac{x}{6}\right)30$$

$$42 = 5x \qquad \text{Divide both sides by } 5.$$

$$x = \frac{42}{5} = 8.4$$

There is a short-cut for solving proportions that avoids calculating an LCD. Observe that we can arrive at the equation $42 = 5x$ directly from the original proportion by *cross-multiplying*:

$$\frac{7}{5} = \frac{x}{6}$$

$$42 = 5x$$

We then proceed as before to complete the solution.

The cross-multiplying shortcut is a fundamental property of proportions.

**Property of Proportions**

If $\dfrac{a}{b} = \dfrac{c}{d}$, then $ad = bc$

**Example 3** The scale on a map of Fairfield County says that $\frac{3}{4}$ centimeter represents a distance of 10 kilometers. If Eastlake and Kenwood are 6 centimeters apart on the map, what is the distance between the two towns?

**Solution** The ratio of the two actual distances is the same as the ratio of the corresponding distances on the map. We let $x$ stand for the distance between Eastlake and Kenwood, and write a proportion. We must be careful to keep the same order in both ratios. We choose to put the distance between towns in the numerators, and the distances on the scale in the denominator.

$$\frac{\text{distance between towns}}{\text{scale}}: \qquad \frac{x}{10} = \frac{6}{\frac{3}{4}}$$

actual map

To solve the propotion, we cross-multiply.

$$\frac{3}{4}x = 10 \cdot 6$$
$$x = \frac{60}{\frac{3}{4}} = 80$$

The two towns are 80 kilometers apart.

**Exercise 3** On a scale model of Fantasy Valley, $1\frac{1}{2}$ inches represents 50 yards. If the distance from the water slide to the bungee jump is 20 inches on the model, what is the distance between the two rides?

**Caution!** Do not try to use "cross-multiplying" on equations that are not proportions, or on any other operations involving fractions. The shortcut works only on proportions.

**RQ1a.** To clear fractions from an equation, we multiply each term by _____ .
**b.** True or False: When clearing fraction from an equation, we do not multiply terms that are not fractions.
**c.** A proportion is a statement that two _____ are equal.
**d.** Cross-multiplying only works on _____ .

## Extraneous Solutions

An algebraic fraction is undefined for any values of $x$ that make its denominator equal zero. These values cannot be solutions to equations involving the fraction. Consider the equation

$$\frac{x}{x-3} = \frac{3}{x-3} + 2$$

When we multiply both sides by the LCD, $(x-3)$, we obtain

$$(x-3)\frac{x}{x-3} = (x-3)\left(\frac{3}{x-3} + 2\right)$$
$$x = 3 + 2x - 6$$

whose solution is $x = 3$. However, $x = 3$ is *not* a solution of the original equation, because both sides of that equation are undefined at $x = 3$. There is no solution to the equation.

What went wrong with our method of solution? We multiplied both sides of the equation by $x - 3$, which is zero when $x = 3$, so we really multiplied both sides of the

equation by zero. Multiplying by zero does not produce an equivalent equation, and false solutions may be introduced. An apparent solution that does not satisfy the original equation is called an **extraneous solution.** Whenever we multiply an equation by an expression containing the variable, we should check that the solution obtained does not cause any of the fractions to be undefined.

**Example 4** Solve the equation $\frac{6}{x}+1=\frac{1}{x+2}$

**Solution** We multiply both sides by the LCD, $x(x+2)$. Note that we multiply *each term* on the left side by the LCD, to get

$$x(x+2)\left(\frac{6}{x}+1\right)=x(x+2)\left(\frac{1}{x+2}\right)$$

or

$$6(x+2)+x(x+2)=x$$

We use the distributive law to remove the parentheses and write the result in standard form:

$$\begin{aligned}6x+12+x^2+2x&=x\\ x^2+7x+12&=0\end{aligned}$$

This is a quadratic equation that we can solve by factoring.

$$(x+3)(x+4)=0$$

so the solutions are $x=-3$ and $x=-4$. Neither of these values causes either denominator to equal zero, so they are not extraneous solutions.

**Caution!** The following "solution" for Example 4 is incorrect. Do you see why?

$$\begin{aligned}x(x+2)\frac{6}{x}+1&=\frac{1}{x+2}x(x+2)\\ 6x+12+1&=x\\ 5x&=-13\\ x&=\frac{-13}{5}\end{aligned}$$

**RQ2a.** An algebraic fraction is undefined when _____.
**b.** An apparent solution that does not satisfy the original equation is called an _____ .
**c.** We may introduce extraneous solutions if we multiply both sides of an equation by___ .
**d.** We must multiply each _____ of the equation by the LCD.

**Exercise 4** Solve $\frac{9}{x^2+x-2}+\frac{1}{x^2-x}=\frac{4}{x-1}$

## Solving Graphically

We can use graphs to solve the equation in Example 4.

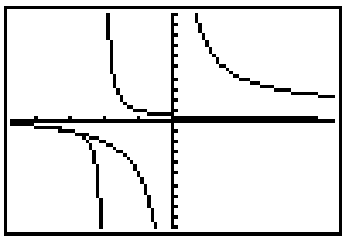
**Example 5** Solve the equation graphically: $\frac{6}{x}+1=\frac{1}{x+2}$

**Solution** We graph the two functions

$$\mathbf{Y_1} = \frac{6}{x} + 1 \quad \text{and} \quad \mathbf{Y_2} = \frac{1}{x+2}$$

in the window

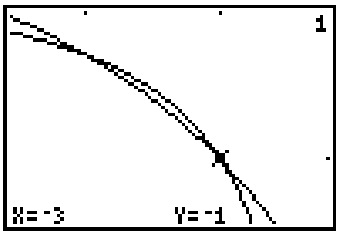
Xmin $= -4.7$ Xmax $= 4.7$
Ymin $= -10$ Ymax $= 10$



as shown at right.

It appears that the two graphs intersect near $x = -3$. To investigate further, we change the window settings to

Xmin $= -4.55$ Xmax $= -2.2$
Ymin $= -1.3$ Ymax $= -0.3$




and obtain the close-up view shown at right. In this window we can see that the graphs intersect in two distinct points, and by using the **Trace** we find that their $x$-coordinates are $x = -3$ and $x = -4$.

**Exercise 5** The manager of a new health club kept track of the number of active members over the club's first few months of operation. The number, $N$, of active members, in hundreds, $t$ months after the club opened is given by the equation

$$N = \frac{10t}{4 + t^2}$$

**a.** Graph the equation in the window

Xmin $= 0$ Xmax $= 9.4$
Ymin $= 0$ Ymax $= 3$

**b.** Use the graph to find out in which months the club had 200 active members.
**c.** Verify your answer algebraically by solving an equation.

## Applications

Motion problems may lead to equations with algebraic fractions.

**Example 6** Rani times herself as she kayaks 30 miles down the Derwent River with the help of the current. Returning upstream against the current she manages only 18 miles in the same amount of time. Rani knows that she can kayak at a rate of 12 miles per hour in still water. What is the speed of the current?

**Solution** If we let $x$ represent the speed of the current, we can fill in the following table.

| | Distance | Rate | Time |
|---|---|---|---|
| Downstream | 30 | $12 + x$ | $\frac{30}{12 + x}$ |
| Upstream | 18 | $12 - x$ | $\frac{18}{12 - x}$ |

Because Rani paddled for equal amounts of time upstream and downstream, we have the equation

$$\frac{30}{12+x} = \frac{18}{12-x}$$

The LCD for the fractions in this equation is $(12+x)(12-x)$. We multiply both sides of the equation by the LCD to obtain

$$(12+x)(12-x)\frac{30}{12+x} = \frac{18}{12-x}(12+x)(12-x)$$
$$30(12-x) = 18(12+x)$$

Solving this equation, we find

$$360 - 30x = 216 + 18x$$
$$144 = 48x$$
$$3 = x$$

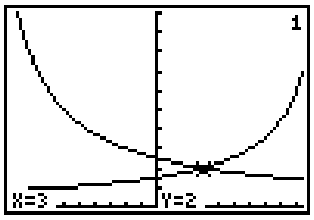
The speed of the current is 3 miles per hour.

We can also solve the equation in Example 6 by graphing two functions, one for each side of the equation. We graph the two functions

$$\mathbf{Y_1} = \frac{30}{12+x} \quad \text{and} \quad \mathbf{Y_2} = \frac{18}{12-x}$$

in the window

Xmin $= -9.4$ Xmax $= 9.4$
Ymin $= 0$ Ymax $= 10$




to obtain the graph shown in the figure. The function $\mathbf{Y_1}$ gives the time it takes Rani to paddle 30 miles downstream, and $\mathbf{Y_2}$ gives the time it takes her to paddle 18 miles upstream. Both of these times depend on the speed of the current, $x$. We are looking for a value of $x$ that makes $\mathbf{Y_1}$ and $\mathbf{Y_2}$ equal. This occurs at the intersection point of the two graphs, $(3, 2)$. Thus, the speed of the current is 3 miles per hour, as we found in Example 6. The $y$-coordinate of the intersection point gives the time Rani paddled on each part of her trip: 2 hours each way.

**Exercise 6** A cruise boat travels 18 miles downstream and back in $4\frac{1}{2}$ hours. If the speed of the current is 3 miles per hour, what is the speed of the boat in still water?

**a.** Let $x$ represent the speed of the boat in still water, and fill in the table.

| | Distance | Rate | Time |
|---|---|---|---|
| Downstream trip | | | |
| Upstream trip | | | |

**b.** Write an equation to model the problem:

Downstream time + Upstream time = Total trip time

**c.** Solve your equation, and answer the question in the problem.

## Formulas

Algebraic fractions may appear in formulas that relate several variables. If we want to solve for one variable in terms of the others, we may need to clear the fractions.

**Example 7** Solve for $P$: $\dfrac{1}{T} = \dfrac{PR}{A-P}$

**Solution** The LCD for the two fractions in the equation is $T(A-P)$. We multiply both sides of the equation by the LCD to obtain

$$\cancel{T}(A-P)\frac{1}{\cancel{T}} = \frac{PR}{A-P}T(A-P)$$

or

$$A - P = PRT$$

Next, we get all the terms containing the desired variable, $P$, on one side of the equation. We add $P$ to both sides to get

$$A = P + PRT$$

We now have two unlike terms that contain the desired variable. To proceed, we *factor out* this variable, and then divide both sides by the remaining factor.

$$A = P(1+RT) \qquad \text{Divide both sides by } 1+RT.$$
$$\frac{A}{1+RT} = \frac{P(1+RT)}{1+RT}$$
$$P = \frac{A}{1+RT}$$

**Exercise 7** Solve for $a$: $\dfrac{2ab}{a+b} = H$

**RQ3** True or False.

**a.** To solve an equation graphically, we graph two functions, $y =$ (each side of the equation.)

**b.** The solutions are the $x$-coordinates of the intersection points of the two graphs.

**c.** To solve a formula that is linear in the desired variable, we must get all terms including that variable on one side of the equation.

**d.** We then factor out the desired variable.

## Answers to Exercises

**1.** $-2, 4$ **2.** $1, 4$ **3.** $666\frac{2}{3}$ yd **4.** $\frac{-1}{2}$ **5a.**

**b.** $1, 4$

**6a.**

| | Distance | Rate | Time |
|---|---|---|---|
| Downstream trip | 18 | $x+3$ | $\frac{18}{x+3}$ |
| Upstream trip | 18 | $x-3$ | $\frac{18}{x-3}$ |

**b.** $\dfrac{18}{x+3} + \dfrac{18}{x-3} = \dfrac{9}{2}$ **c.** 9 mph

**7.** $a = \dfrac{bH}{2b-H}$

**Now go to WeBWorK and do the Reading Assignment for this Lesson.**

# Homework 39

**1.** A small lake in a state park has become polluted by runoff from a factory upstream. The cost for removing $p$ percent of the pollution from the lake is given, in thousands of dollars, by

$$C = \frac{25p}{100 - p}$$

How much of the pollution can be removed for $25,000?

**2.** A school of bluefin tuna is making a 200 mile trip at an average air speed of 36 miles per hour in still water, but is swimming against a current.

**a.** Express the tuna's travel time, $t$, as a function of the current speed, $v$, and graph the function on the window

$$\text{Xmin} = 0 \quad \text{Xmax} = 36$$
$$\text{Ymin} = 0 \quad \text{Ymax} = 50$$

**b.** Write and solve an equation to find the current speed if the school makes its trip in $8$ hours. Label the corresponding point on your graph.

**3.** During the baseball season so far this year, Pete got hits 34 times out of 164 times at bat.

**a.** What is Pete's batting average so far? (Batting average is the fraction of at-bats that resulted in hits.)

**b.** If Pete gets hits on every one of his next $x$ at-bats, write an expression for his new batting average.

**c.** How many consecutive hits does Pete need to raise his batting average to $0.350$?

☐ **a.** Solve the equation graphically by graphing two functions, one for each side of the equation.

**b.** Solve the equation algebraically.

**4.** $\dfrac{2x}{x+1} = \dfrac{x+1}{2}$ **5.** $\dfrac{2}{x+1} = \dfrac{x}{x+1} + 1$ **6.** $\dfrac{15x}{1+x^2} = 6$

**7.** The manager of Joe's Burgers discovers that he will sell $\dfrac{160}{x}$ burgers per day if the price of a burger is $x$ dollars. On the other hand, he can afford to make $6x + 49$ burgers if he charges $x$ dollars apiece for them.

**a.** Graph the demand function, $D(x) = \dfrac{160}{x}$, and the supply function, $S(x) = 6x + 49$, in the same window. At what price $x$ does the demand for burgers equal the number that Joe can afford to supply? This value for $x$ is called the equilibrium price.

**b.** Write and solve an equation to verify your equilibrium price.

**8.** Compare the methods in each calculation with fractions. Explain how the LCD is used in each operation.

**a.** Add: $\dfrac{3}{x} + \dfrac{1}{x+3}$ **b.** Divide: $\dfrac{3}{x} \div \dfrac{1}{x+3}$

**c.** Solve: $\dfrac{3}{x} + \dfrac{1}{x+3} = 3$ **d.** Simplify: $\dfrac{1 + \frac{3}{x}}{3 + \frac{1}{x+3}}$

**9.** Find the error in the following "proof" that $1 = 0$:
Start by letting $x = 1$.

$$\begin{aligned}
x &= 1 && \text{Multiply both sides by } x.\\
x^2 &= x && \text{Subtract } 1 \text{ from both sides.}\\
x^2 - 1 &= x - 1 && \text{Factor the left side.}\\
(x-1)(x+1) &= x - 1 && \text{Divide both sides by } x - 1.\\
\frac{(x-1)(x+1)}{x-1} &= \frac{x-1}{x-1} && \text{Simplify both sides.}\\
x + 1 &= 1 && \text{Subtract } 1 \text{ from both sides.}\\
x &= 0
\end{aligned}$$

Because $x = 1$ and $x = 0$, we have "proved" that $1 = 0$.

**10.** What is wrong with the following addition problem? Add $\frac{3}{8} + \frac{1}{2}$

Solution:
$$\begin{aligned}
8 \cdot \frac{3}{8} + 8 \cdot \frac{1}{2} &= 3 + 4 && \text{Multiply by the LCD, } 8.\\
&= 7 && \text{The answer is } 7.
\end{aligned}$$

☐ Solve each formula for the specified variable.

**11.** $S = \frac{a}{1-r}$ for $r$

**12.** $m = \frac{y-k}{x-h}$ for $x$

**13.** $\frac{x}{a} + \frac{y}{b} = 1$ for $x$

**14.** $F = \frac{Gm_1m_2}{d^2}$ for $d$

**15.** $C = \frac{rR}{r+R}$ for $R$

**16.** $E = \frac{Ff}{(P-x)p}$ for $x$

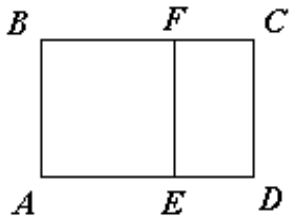
**17.** A chartered sight-seeing flight over the Grand Canyon is scheduled to return to its departure point in 3 hours. The pilot would like to cover a distance of 144 miles before turning around, and he hears on the weather service that there will be a headwind of 20 miles per hour on the outward journey.

- **a.** Express the time it takes for the outward journey as a function of the air speed of the plane.
- **b.** Express the time it takes for the return journey as a function of the speed of the plane
- **c.** Graph the sum of the two functions and find the point on the graph with $y$-coordinate $3$. Interpret the coordinates of the point in the context of the problem.
- **d.** The pilot would like to know what air speed to maintain in order to complete the tour in 3 hours. Write an equation to describe this situation.
- **e.** Solve your equation to find the appropriate air speed.

**18.** The cost of wire fencing is $7.50 per foot. A rancher wants to enclose a rectangular pasture of 1000 square feet with this fencing.

- **a.** Express the length of the pasture as a function of its width.
- **b.** Express the cost of the fence as a function of its width.
- **c.** Graph your function for the cost and find the coordinates of the lowest point on the graph. Interpret those coordinates in the context of the problem.
- **d.** The rancher has $1050 to spend on the fence, and she would like to know what the width of the pasture should be. Write an equation to describe this situation.
- **e.** Solve your equation and find the dimensions of the pasture.

**19.** The rectangle $ABCD$ is divided into a square and a smaller rectangle, $CDEF$. The two rectangles $ABCD$ and $CDEF$ are similar. A rectangle $ABCD$ with this property is called a golden rectangle, and the ratio of its length to its width is called the golden ratio. The golden ratio appears frequently in art and nature, and is considered to give the most pleasing proportions to many figures. We'll compute the golden ratio as follows.

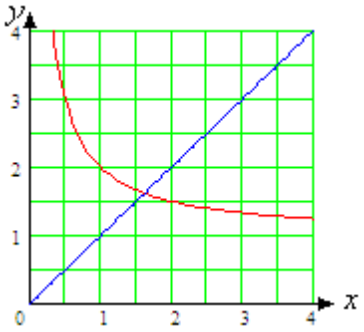



**a.** Let $AB = 1$ and $AD = x$. What are the lengths of $AE$, $ED$, and $CD$?

**b.** Write a proportion in terms of $x$ for the similarity of rectangles $ABCD$ and $CDEF$. Be careful to match up the corresponding sides.

**c.** Solve your proportion for $x$. Find the golden ratio, $\frac{AD}{AB} = \frac{x}{1}$.

**20.** The figure shows the graphs of two equations, $y = x$ and $y = \frac{1}{x} + 1$.




**a.** Find the $x$-coordinate of the intersection point of the two graphs.

**b.** Compare your answer to the golden ratio you computed in Problem 19.

**Now go to WeBWorK and do the Skills Practice for this Homework.**