

# Lesson 23 Integer Exponents

You know that a positive integer exponent tells us how many copies of its base to multiply together. For example,

$$4a^3b^2 \quad \text{means} \quad 4aaabb$$

What is the meaning of a negative exponent?

## Negative Exponents

Study the list of powers of 2 shown in Table A, and observe the pattern as we move up the list from bottom to top. Each time the exponent increases by 1 we multiply by another factor of 2. We can continue up the list as far as we like.

If we move back down the list, we divide by 2 at each step, until we get to the bottom of the list,  $2^1 = 2$ . What if we continue the list in the same way, dividing by 2 each time we decrease the exponent? The results are shown in Table B.

As we continue to divide by 2, we generate fractions whose denominators are powers of 2. In particular,

$$2^{-1} = \frac{1}{2} = \frac{1}{2^1} \quad \text{and} \quad 2^{-2} = \frac{1}{4} = \frac{1}{2^2}$$

Based on these observations, we make the following definitions.

### Negative and Zero Exponents

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$$

$$a^0 = 1 \quad (a \neq 0)$$

As long as the base  $a$  is not zero, a negative exponent denotes a reciprocal.

**Example 1a.**  $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$       **b.**  $9x^{-2} = 9 \cdot \frac{1}{x^2} = \frac{9}{x^2}$

### Caution!

1. A negative exponent does *not* mean that the power is negative! For example,

$$2^{-3} \neq -2^3$$

2. In Example 1b, note that

$$9x^{-2} \neq \frac{1}{9x^2}$$

The exponent,  $-2$ , applies *only* to the base  $x$ , not to 9.

⋮

$$2^4 = 16$$

$$8 \times 2 = 16$$

$$2^3 = 8$$

$$4 \times 2 = 8$$

$$2^2 = 4$$

$$2 \times 2 = 4$$

$$2^1 = 2$$

**Table A**

⋮

$$2^3 = 8$$

$$8 \div 2 = 4$$

$$2^2 = 4$$

$$4 \div 2 = 2$$

$$2^1 = 2$$

$$2 \div 2 = 1$$

$$2^0 = 1$$

$$1 \div 2 = \frac{1}{2}$$

$$2^{-1} = \frac{1}{2}$$

$$\frac{1}{2} \div 2 = \frac{1}{4}$$

$$2^{-2} = \frac{1}{4}$$

⋮

**Table B**

## 2 Unit 6 Powers and Roots

**RQ1** True or False.

- a. Negative exponents are used to denote reciprocals.
- b. We can write  $\frac{1}{4x^3}$  as  $4x^{-3}$ .
- c. Any number raised to the zero power is zero.
- d.  $(-4)^{-3} = 4^3$ .

**Exercise 1** Write each expression without using negative exponents.

a.  $5^{-4}$

b.  $5x^{-4}$

In the next example, we evaluate an expression that contains negative exponents, and solve an equation involving negative exponents.

**Example 2** The body mass index, or BMI, is one measure of a person's physical fitness. Your body mass index is defined by

$$\text{BMI} = wh^{-2}$$

where  $w$  is your weight in kilograms and  $h$  is your height in meters. The World Health Organization classifies a person as obese if his or her BMI is 25 or higher.

- a. Calculate the BMI for a woman who is 1.625 meters (64 inches) tall and weighs 54 kilograms (120 pounds).
- b. For a fixed weight, how does BMI vary with height?
- c. The world's heaviest athlete is the amateur sumo wrestler Emanuel Yarbrough, who weighs 319 kg (704 pounds). What height would Yarbrough have to be to have a BMI under 25?

**Solutions** a. We evaluate the formula to find

$$\text{BMI} = 54(1.625^{-2}) = 54\left(\frac{1}{1.625^2}\right) = 20.45$$

b.  $\text{BMI} = wh^{-2} = \frac{w}{h^2}$ , so BMI varies inversely with the square of height. Thus, for a fixed weight, BMI decreases as height increases.

c. To find the height that gives a BMI of 25, we solve the equation  $25 = 319h^{-2}$ . The variable  $h$  appears in the denominator of a fraction, so we begin by clearing the denominator — we multiply both sides of the equation by  $h^2$ .

$$\begin{array}{ll} 25 = \frac{319}{h^2} & \text{Multiply both sides by } h^2. \\ 25h^2 = 319 & \text{Divide both sides by 25.} \\ h^2 = 12.76 & \text{Extract square roots.} \\ h \approx 3.57 \end{array}$$

To have a BMI under 25, Yarbrough would have to be over 3.57 meters, or 11 feet 8 inches tall. (In fact, he is 6 feet 8 inches tall.)

**Exercise 2** Solve the equation  $0.2x^{-3} = 1.5$ .

Rewrite without a negative exponent.

Clear the fraction.

Isolate the variable.

## Power Functions

The functions that describe direct and inverse variation are part of a larger family of functions called **power functions**.

### Power Functions

A function of the form

$$f(x) = kx^p$$

where  $k$  and  $p$  are non-zero constants, is called a **power function**.

Examples of power functions are

$$V(r) = \frac{4}{3}\pi r^3 \quad \text{and} \quad L(T) = 0.8125 T^2$$

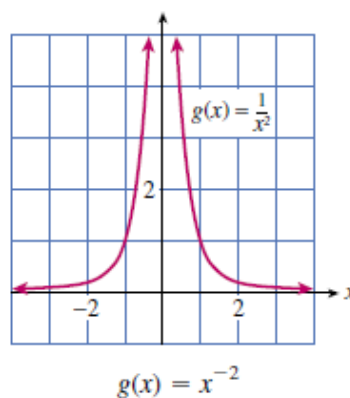
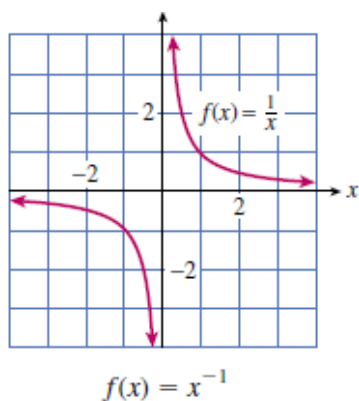
In addition, the basic functions

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = \frac{1}{x^2}$$

can be written as

$$f(x) = x^{-1} \quad \text{and} \quad g(x) = x^{-2}$$

Their graphs are shown below. Note that power functions with negative exponents are not defined at zero.



**Example 3** Which of the following are power functions?

- a.  $f(x) = \frac{1}{3}x^4 + 2$    b.  $g(x) = \frac{1}{3x^4}$    c.  $h(x) = \frac{x+6}{x^3}$

**Solutions** a. This is not a power function because of the addition of the constant term.

b. We can write  $g(x) = \frac{1}{3}x^{-4}$ , so  $g$  is a power function.

c. This is not a power function, but it can be treated as the sum of two power functions, because  $h(x) = x^{-2} + 6x^{-3}$ .

**RQ2** True or False.

- The exponent in a power function must be positive.
- Direct and inverse variation are examples of power functions.
- If  $f$  is a power function, then  $f(0) = 0$ .
- The sum of two power functions is not always a power function.

## 4 Unit 6 Powers and Roots

**Exercise 3** Write each function as a power function in the form  $y = kx^p$ .

a.  $f(x) = \frac{12}{x^2}$

b.  $g(x) = \frac{1}{4x}$

c.  $h(x) = \frac{2}{5x^6}$

Most applications are concerned with positive variables only, so many models use only the portion of the graph in the first quadrant.

**Example 4** The intensity of the radiation from our Sun varies inversely with distance.

a. Use a negative exponent to write a formula for the intensity  $I$  as a function of  $d$ ,  $I = f(d)$ .

b. The planet Mercury is 0.387 AU (Astronomical Units) distant from the Sun, and the intensity of radiation at its surface is 9147 watts per square meter. Find the constant of proportionality in the formula for  $I$ .

c. Graph  $I = f(d)$ .

d. Earth is 1 AU from the sun. What is the intensity of the Sun's radiation at the surface of the Earth?

e. The surface of Jupiter receives 50.63 watts per square meter of the Sun's radiation. How far is Jupiter from the Sun?

**Solutions a.** If we use  $k$  for the constant of proportionality, then  $I = \frac{k}{d^2}$ . Rewriting this equation with a negative exponent gives  $I = kd^{-2}$ .

b. We substitute  $I = 9147$  and  $d = 0.387$  to obtain

$$\begin{aligned} 9147 &= k(0.387)^{-2} && \text{Multiply both sides by } 0.387^2. \\ 1369.9 &= k \end{aligned}$$

Thus,  $I = f(d) = 1370 d^{-2}$ .

c. We evaluate the function for several values of  $d$ , and use a calculator to obtain the graph shown.

d. We substitute  $d = 1$  into the formula to obtain

$$I = 1370(1)^{-2} = 1370$$

Earth receives 1370 watts/m<sup>2</sup> of solar radiation.

e. We substitute  $I = 50.63$  into the formula, and solve for  $d$ .

$$\begin{aligned} 50.63 &= 1370 d^{-2} = \frac{1370}{d^2} && \text{Multiply both sides by } d^2. \\ 50.63 d^2 &= 1370 && \text{Divide both sides by } 50.63. \\ d^2 &= \frac{1370}{50.63} = 27.059 && \text{Extract roots.} \\ d &= 5.202 \end{aligned}$$

Jupiter is 5.202 AU from the sun, or about 484 million miles.



The function  $I = \frac{k}{d^2}$  is an example of an **inverse square law**, because  $I$  varies inversely with the square of  $d$ . Such laws are common in physics and its applications, because gravitational and other forces behave in this way. Here is another example of an inverse square law.

**Exercise 4** Cell phone towers typically transmit signals at 10 watts of power. The signal strength varies inversely with the square of distance from the tower, and 1 kilometer away the signal strength is 0.8 picowatt. (A picowatt is  $10^{-12}$  watt.) Cellphones can receive a

signal as small as 0.01 picowatt. How far can you be from the nearest tower and still hope to have cell phone reception?

## Working with Negative Exponents

The laws of exponents apply to all integer exponents, positive, negative, and zero. When we allow negative exponents, we only need one version of the rule for computing quotients of powers, namely

$$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$$

For example, by applying this new version of the law we find

$$\frac{x^2}{x^5} = x^{2-5} = x^{-3}$$

which is consistent with our previous version of the rule,

$$\frac{x^2}{x^5} = \frac{1}{x^{5-2}} = \frac{1}{x^3}$$

For reference, we restate the laws of exponents below. The laws are valid for all integer exponents  $m$  and  $n$ , and for  $a, b \neq 0$ .

### Laws of Exponents

I.  $a^m \cdot a^n = a^{m+n}$

III.  $(a^m)^n = a^{m \cdot n}$

II.  $\left(\frac{a^m}{a^n}\right) = a^{m-n}$

IV.  $(a \cdot b)^n = a^n \cdot b^n$

V.  $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

**Example 5a.**  $x^3 \cdot x^{-5} = x^{3-5} = x^{-2}$

Apply the first law: add exponents.

b.  $\frac{8x^{-2}}{4x^{-6}} = \frac{8}{4}x^{-2-(-6)} = 2x^4$

Apply the second law: subtract exponents.

c.  $(5x^{-3})^{-2} = 5^{-2}(x^{-3})^{-2} = \frac{1}{25}x^6$

Apply laws IV and III.

You can check that each of the calculations in Example 7 is shorter when we use negative exponents instead of converting the expressions into fractions.

**Exercise 5** Simplify by applying the laws of exponents.

a.  $(2a^{-4})(-4a^2)$

b.  $\frac{(r^2)^{-3}}{3r^{-4}}$

**Caution!** The laws of exponents do not apply to sums or differences of powers. We can add or subtract *like terms*, that is powers with the same exponent. For example,

$$6x^{-2} + 3x^{-2} = 9x^{-2}$$

but **we cannot add or subtract terms with different exponents.** Thus, for example,

$$4x^2 - 3x^{-2} \quad \text{cannot be simplified,}$$

$$x^{-1} + x^{-3} \quad \text{cannot be simplified.}$$

## 6 Unit 6 Powers and Roots

In the opening table we saw that  $2^0 = 1$ , and in fact  $a^0 = 1$  as long as  $a \neq 0$ . Now we can see that this definition is consistent with the laws of exponents. The quotient of any (nonzero) number divided by itself is 1. But by applying the second law of exponents, we also have

$$1 = \frac{a^m}{a^m} = a^{m-m} = a^0$$

### The Zero Exponent

$$a^0 = 1, \quad \text{if } a \neq 0$$

For example,

$$3^0 = 1, \quad (-528)^0 = 1, \quad \text{and} \quad (0.024)^0 = 1$$

**RQ3** True or False.

- a. We cannot add or subtract terms with different exponents.
- b.  $(a + b)^5 = a^5 + b^5$  is an application of the fourth law of exponents.
- c. If  $x \neq 0$ , then  $\frac{k}{x^{-n}} = kx^n$ .
- d.  $(x^2)^{-3} = (x^{-2})^3$  because  $\frac{1}{x^2 \cdot x^2 \cdot x^2} = \left(\frac{1}{x^2}\right)\left(\frac{1}{x^2}\right)\left(\frac{1}{x^2}\right)$ .

## Review of Scientific Notation

Scientists and engineers regularly encounter very large numbers such as

5, 980, 000, 000, 000, 000, 000, 000


(the mass of the earth in kilograms) and very small numbers such as

0.000 000 000 000 000 000 000 001 67

(the mass of a hydrogen atom in grams) in their work. These numbers are easier to use when expressed in **scientific notation**.

### To Write a Number in Scientific Notation:

1. Locate the decimal point so that there is exactly one nonzero digit to its left.
2. Count the number of places you moved the decimal point: this determines the power of 10.
  - a. If the original number is greater than 10, the exponent is positive.
  - b. If the original number is less than 1, the exponent is negative.

 **Example 6** Write each number in scientific notation.

a.  $478,000 = 4.78000 \times 10^5$     *Move the decimal 5 places to the left.*  
 $= 4.78 \times 10^5$

b.  $0.00032 = 00003.2 \times 10^{-4}$     *Move the decimal 4 places to the right.*  
 $= 3.2 \times 10^{-4}$



**Exercise 6** Write each number in scientific notation.

a. 0.063

b. 1480

Your calculator displays numbers in scientific notation if they are too large or too small to fit in the display screen. Try squaring the number 123,456,789 on your calculator. Enter

$$123456789 \quad [x^2]$$

and the calculator will display the result as

$$1.524157875 \text{ E } 16$$

This is how the calculator displays the number  $1.524157875 \times 10^{16}$ . Notice that the power  $10^{16}$  is displayed as **E 16**.

To enter a number in scientific form, we use the key labeled **EE**. For example, to enter  $3.26 \times 10^{-18}$ , we enter the keying sequence

$$3.26 \quad [\text{EE}] \quad [(-)] \quad 18$$

**Example 7** The average American eats 110 kilograms of meat per year. It takes about 16 kilograms of grain to produce one kilogram of meat, and advanced farming techniques can produce about 6000 kilograms of grain on each hectare of arable land. (A *hectare* is 10,000 square meters, or just under two and a half acres.) Now, the total land area of the Earth is about 13 billion hectares, but only about 11% of that land is arable. Is it possible for each of the 6.7 billion people on Earth to eat as much meat as Americans do?

**Solution** First we'll compute the amount of meat necessary to feed every person on earth 110 kilograms per year. In scientific notation, the population of Earth is  $6.7 \times 10^9$  people.

$$(6.7 \times 10^9 \text{ people}) \times (110 \text{ kg/person}) = 7.37 \times 10^{11} \text{ kg meat}$$

Next we'll compute the amount of grain needed to produce that much meat.

$$(16 \text{ kg grain/kg meat}) \times (7.37 \times 10^{11} \text{ kg meat}) = 1.18 \times 10^{13} \text{ kg grain}$$

Next we'll see how many hectares of land are needed to produce that much grain.

$$(1.18 \times 10^{13} \text{ kg grain}) \div (6000 \text{ kg grain/hectare}) = 1.97 \times 10^9 \text{ hectares}$$

Finally, we'll compute the amount of land on Earth suitable for grain production.

$$0.11 \times (13 \times 10^9 \text{ hectares}) = 1.43 \times 10^9 \text{ hectares}$$

The amount of arable land on Earth is less than the amount needed to produce that much grain. Thus, even if we use every hectare of arable land to produce grain for livestock, we won't have enough to provide every person on Earth with 110 kilograms of meat per year.

## Answers to Exercises

- 1a.  $\frac{1}{5^4}$     b.  $\frac{5}{x^4}$     2. 0.513a.  $f(x) = 12x^{-2}$     b.  $g(x) = \frac{1}{4}x^{-1}$   
 c.  $h(x) = \frac{2}{5}x^{-6}$     4. 8.9 km    5a.  $-8a^{-2}$     b.  $\frac{1}{3}r^{-2}$   
 6a.  $6.3 \times 10^{-2}$     b.  $1.48 \times 10^3$

Now go to WeBWork and do the Reading Assignment for this Lesson.

## Homework 23

1. Let  $g(x) = x^{-3}$ . Complete the tables.

a.

$x$	1	2	4.5	6.2	9.3
$g(x)$					

- b. What happens to the values of  $g(x)$  as the values of  $x$  increase? Explain why.

c.

$x$	1.5	0.6	0.1	0.03	0.002
$g(x)$					

- d. What happens to the values of  $g(x)$  as the values of  $x$  decrease toward 0? Explain why.

2. a. Use your calculator to graph the following functions in the window

$$X_{\min} = -5 \quad X_{\max} = 5$$

$$Y_{\min} = -2 \quad Y_{\max} = 10$$

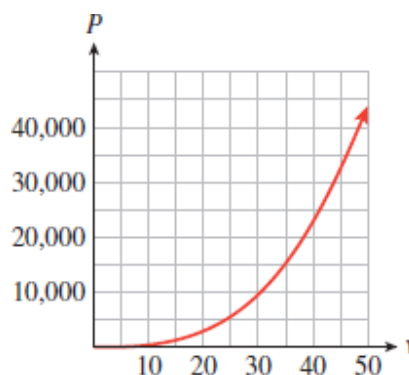
- i.  $f(x) = x^2$       ii.  $f(x) = x^{-2}$       iii.  $f(x) = \frac{1}{x^2}$       iv.  $f(x) = \left(\frac{1}{x}\right)^2$

- b. Which functions have the same graph? Explain your results.

- A** 3. When an automobile accelerates, the power,  $P$ , needed to overcome air resistance varies directly with a power of the speed,  $v$ .

- a. Use the data and the graph to find the scaling exponent and the constant of variation. Then write a formula for  $P$  as a power function of  $v$ .

$v$ (mph)	10	20	30	40
$P$ (watts)	355	2840	9585	22,720



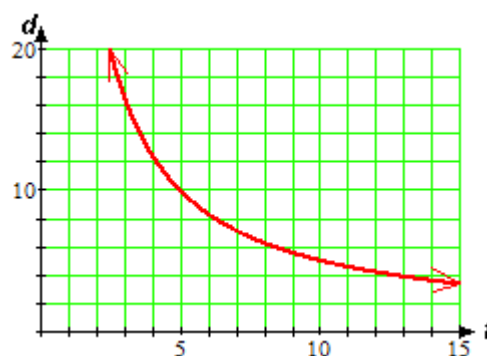
- b. Find the speed that requires 50,000 watts of power.  
 c. If you double your speed, by what factor does the power requirement increase?  
 d. If you increase your speed by 50%, by what factor does the power requirement increase?



4. Poiseuille's law for the flow of liquid through a tube can be used to describe blood flow through an artery. The rate of flow  $F$  in liters per minute is proportional to the fourth power of the radius,  $r$ , divided by the length,  $L$ , of the artery.
- Write a formula for the rate of flow as a power function of radius.
  - If the radius and length of the artery are measured in centimeters, then the constant of variation,  $k = 7.8 \times 10^5$ , is determined by blood pressure and viscosity. If a certain artery is 20 centimeters long, what should its radius be in order to allow a blood flow of 5 liters per minute?

- A** 5. The  $f$ -stop setting on a camera regulates the size of the aperture, and thus the amount of light entering the camera. The diameter,  $d$ , of the aperture is inversely proportional to the  $f$ -stop.
- Use the data and the graph to find the constant of proportionality and write  $d$  as a power function of  $f$ . Values of  $d$  have been rounded to one decimal place.

$f$	2.8	4	5.6	8	11
$d$	17.9	12.5	8.9	6.3	4.5



- Why are the  $f$ -stop settings labeled with the values given in the table? (Hint: The amount of light entering the camera is proportional to the area of the aperture. Calculate  $r^2$ , where  $r$  is the radius of the aperture, for each value of  $d$ , and round to the nearest whole number.)
6. The lifetime of a star is roughly inversely proportional to the cube of its mass. Our Sun, which has a mass of 1 solar mass, will last for approximately 10 billion years.
- Write a power function for the lifetime,  $L$ , of a star in terms of its mass,  $m$ .
  - Sketch a graph of the function, using units of solar mass on the horizontal axis.
  - How long will a star that is ten times more massive than the Sun last?
  - One solar mass is about  $2 \times 10^{30}$  kilograms. Rewrite your formula for  $L$  with the units of mass in kilograms.
  - How long will a star that is half as massive as the Sun last?

**Now go to WeBWork and do the Skills Practice for this Homework.**