

Lesson 18 Functions

Definitions and Notation

We often want to predict the values of one variable from the values of a related variable. For example, when a physician prescribes a drug, she needs to know how long each dose will remain in the bloodstream. A sales manager needs to know how the price of his product will affect his sales. A function is a special type of relationship between variables that allows us to make such predictions.

We have already seen some examples of functions. For instance, suppose it costs \$800 for flying lessons plus \$30 per hour to rent a plane. If we let C represent the total cost for t hours of flying lessons, then

$$C = 800 + 30t \quad (t \geq 0)$$

The variable t is called the **input** variable, and C is the **output** variable. Given a value of the input, we can calculate the corresponding output value using the formula for the function.

$$\text{when } t = 0, \quad C = 800 + 30(0) = 800$$

$$\text{when } t = 4, \quad C = 800 + 30(4) = 920$$

$$\text{when } t = 10, \quad C = 800 + 30(10) = 1100$$

We can display the relationship between two variables by a table or by ordered pairs. The input is the first component of the ordered pair, and the output is the second component. For the example above we have:

t	C	(t, C)
0	800	(0, 800)
4	920	(4, 920)
10	1100	(10, 1100)

Notice that only *one* value of C corresponds to each value of t . We say that " C is a **function** of t ."

Definition of Function

A **function** is a relationship between two variables for which a *unique* value of the **output** variable is assigned to each value of the **input** variable.

RQ1 What distinguishes functions from other variable relationships?

- a. Each value of the input variable corresponds to exactly one value of the output variable.
- b. We can display the variables as ordered pairs.
- c. The variables are related by a formula.
- d. The values of the input and output variables must be different.

■ **Example 1a.** The distance, d , traveled by a car in 2 hours is a function of its speed, r . If we know the speed of the car, we can find the distance it travels by using the formula $d = r \cdot 2$.

b. The price of a fill-up with unleaded gasoline is a function of the number of gallons you buy. The gas pump represents the function by displaying the corresponding values of the input variable (number of gallons) and the output variable (total price).

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c. Score on the Scholastic Aptitude Test is *not* a function of score on an IQ test, because two people with the same score on an IQ test may score differently on the SAT; that is, a person's score on the SAT is not uniquely determined by his or her score on an IQ test. ■

Exercise 1a. As part of a project to improve the success rate of freshmen, the counselling department studied the grades earned by a group of students in English and algebra. Do you think that a student's grade in algebra is a function of his or her grade in English? Explain why or why not.

b. Phatburger features a soda bar where you can serve your own soft drinks in any size. Do you think that the number of calories in a serving of Zap Kola is a function of the number of fluid ounces? Explain why or why not.

Functions Defined by Tables

A function can be described in several different ways. In the following examples we consider functions defined by tables, by graphs, and by equations.

■ **Example 2a.** The table shows data on sales compiled over several years by Eau Claire Auto Parts.

Year (t)	Total Sales (S)
1990	\$612,000
1991	\$663,000
1992	\$692,000
1993	\$749,000
1994	\$904,000

In this example, the year is the input variable, and total sales is the output. We say that the total sales, S , is a function of t .

b. This table gives the postage for first-class mail in 2008.

Weight in Ounces (w)	Postage (p)
$0 < w \leq 1$	\$0.41
$1 < w \leq 2$	\$0.58
$2 < w \leq 3$	\$0.75
$3 < w \leq 3.5$	\$0.92
$3.5 < w \leq 4$	\$1.31
$4 < w \leq 5$	\$1.48
$5 < w \leq 6$	\$1.65

If we know the weight of an item, we can determine the required postage from the table. For instance, a catalog that weighs 4.5 ounces costs \$1.48 in postage. In this example, w is the input variable and p is the output. We say that p is a function of w .

c. The table on the next page records the age and cholesterol count for 20 patients tested in a hospital survey. According to these data, cholesterol count is *not* a function of age, because several patients who are the same age have different cholesterol levels. For example, patients 316, 332, and 340 are all 51 years old, but they have cholesterol counts of 227, 209, and 216, respectively. We cannot determine a *unique* value of the output variable (cholesterol count) from the value of the input variable (age). Other factors besides age must influence a person's cholesterol count.

Patient Number	Age	Cholesterol Count	Patient Number	Age	Cholesterol Count
301	53	217	332	51	209
308	48	232	336	53	241
312	55	198	339	49	186
313	56	238	340	51	216
316	51	227	343	57	208
320	52	264	347	52	248
322	53	195	356	50	214
324	47	203	359	56	271
325	48	212	362	53	193
328	50	234	370	48	172

Exercise 2 Decide whether each table describes y as a function of x . Explain your choice.

a.

x	3.5	2.0	2.5	3.5	2.5	4.0	2.5	3.0
y	2.5	3.0	2.5	4.0	3.5	4.0	2.0	2.5

b.

x	-3	-2	-1	0	1	2	3
y	17	3	0	-1	0	3	17

RQ2 Which statements about functions are true?

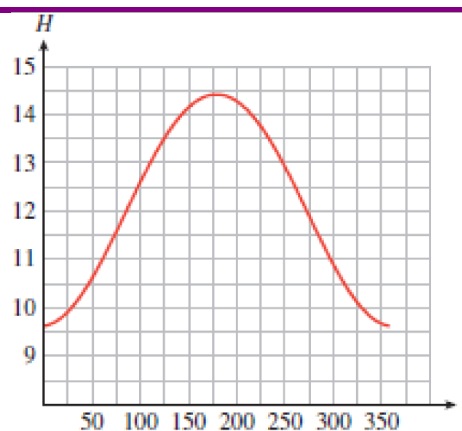
- The first row or column of a table represents the input variable.
- There cannot be two inputs with the same output.
- There cannot be two outputs with the same input.
- There must be a pattern to the output values.

Functions Defined by Graphs

We can also use a graph to define a function.

Example 3 The graph shows the number of hours of sunlight, H , in Peoria, Illinois on day t , where $t = 0$ on January 1.

- Which variable is the input, and which is the output?
- How many hours of sunlight are there in Peoria on day 150?
- On which days are there 12 hours of sunlight?
- What are the maximum and minimum values of H , and when do these values occur?



Solutions a. The input variable, t , appears on the horizontal axis. The number of daylight hours, H , depends on the date. The output variable appears on the vertical axis.

b. The point on the curve where $t = 150$ has $H \approx 14.1$, so Peoria has about 14.1 hours of daylight when $t = 150$, which is at the end of May.

c. $H = 12$ at two points, where $t \approx 85$ (in late March) and $t \approx 270$ (late September).

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d. The maximum value of 14.4 hours is the longest day of the year, when $t \approx 170$, about three weeks into June. The minimum of 9.6 hours is the shortest day, when $t \approx 355$, about three weeks into December.

Exercise 3 The graph shows the elevation in feet, a , of the Los Angeles Marathon course at a distance d miles into the race. (Source: Los Angeles Times, March 3, 2005)



- Which variable is the input, and which is the output?
- What is the elevation at mile 20?
- At what distances is the elevation 150 feet?
- What are the maximum and minimum values of a , and when do these values occur?
- The runners pass by the Los Angeles Coliseum at about 4.2 miles into the race. What is the elevation there?

Functions Defined by Equations

Example 4 illustrates a function defined by an equation.

Example 4 As of 2008, the Sears Tower in Chicago is the nation's tallest building at 1454 feet. If an algebra book is dropped from the top of the Sears Tower, its height above the ground after t seconds is given by the equation

$$h = 1454 - 16t^2$$

Thus, after 1 second the book's height is

$$h = 1454 - 16(1)^2 = 1438 \text{ feet}$$

after 2 seconds its height is

$$h = 1454 - 16(2)^2 = 1390 \text{ feet}$$

For this function, t is the input variable and h is the output variable. For any value of t , we can find a unique value of h from the equation. We say that h is a function of t .

Exercise 4 Write an equation that gives the volume, V , of a sphere as a function of its radius, r .

RQ3 The rule relating the input and output variables of a function may be described in words. Name three more ways that a function can be described.

Function Notation

There is a convenient notation for discussing functions. First, we choose a letter, such as f , g , or h , (or F , G , or H), to name the function. (We can use any letter, but these are the most common choices.) For instance, in Example 4, the height h of a falling algebra book is a function of the elapsed time, t . We might call this function f . We write

$$h = f(t)$$

which means that h is a function of t , and f is the name of the function. The symbol $f(t)$, read " f of t ," is another name for the height h .

Caution! The parentheses in the symbol $f(t)$ do *not* indicate multiplication. (It does not make sense to multiply the name of a function by a variable.) Think of the symbol $f(t)$ as a single variable that represents the output value of the function. ■

With this new notation we may write

$$h = f(t) = 1454 - 16t^2$$

or just

$$f(t) = 1454 - 16t^2$$

instead of

$$h = 1454 - 16t^2$$

to describe the function.

It may seem complicated to introduce a new symbol for h , but the notation $f(t)$ is very useful for showing the correspondence between specific values of the variables h and t .

■ **Example 5** In Example 4, the height of an algebra book dropped from the top of the Sears Tower is given by the equation

$$h = 1454 - 16t^2$$

We see that

$$\text{when } t = 1, \quad h = 1438$$

$$\text{when } t = 2, \quad h = 1390$$

Using function notation, these relationships can be expressed more concisely as

$$f(1) = 1438 \quad \text{and} \quad f(2) = 1390$$

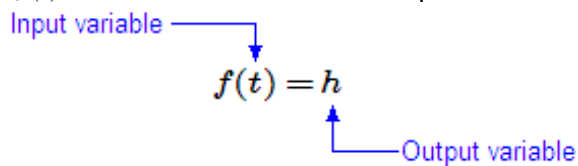
which we read as " f of 1 equals 1438" and " f of 2 equals 1390." The values for the input variable, t , appear inside the parentheses, and the values for the output variable, h , appear on the other side of the equation. ■

RQ4 True or False.

- The notation $f(t)$ indicates the product of f and t .
- If $y = f(x)$, then $f(x)$ gives the value of the input variable.
- If Q is a function of M , we may write $M = f(Q)$.
- In the equation $d = g(n)$, the letters d , g , and n are variables.

Function Notation

The symbol $f(t)$ is another name for the output variable of a function.



Exercise 5 Let F be the name of the function defined by the graph in Example 3, the number of hours of sunlight in Peoria.

- Use function notation to state that H is a function of t .
- What does the statement $F(15) = 9.7$ mean in the context of the problem?

Using Function Notation

Finding the value of the output variable that corresponds to a particular value of the input variable is called **evaluating** the function.

Example 6 Let g be the name of the postage function Example 2b. Find $g(1)$, $g(3)$, and $g(5.75)$.

Solution According to the table,

when $w = 1$, $p = 0.41$, so $g(1) = 0.41$
 when $w = 3$, $p = 0.75$, so $g(3) = 0.75$
 when $w = 5.75$, $p = 1.65$, so $g(5.75) = 1.65$

Thus, a letter weighing 1 ounce costs \$0.41 to mail; a letter weighing 3 ounces costs \$0.75, and a letter weighing 5.75 ounces costs \$1.65.

We can also find the input (or inputs) corresponding to a given output. For example, if $p = g(w)$ is the postage function, we solve the equation $g(w) = 0.58$ by finding all input values, w , that correspond to the output \$0.58. According to the table in Example 2b, any value of w greater than 1 but less than or equal to 2 is a solution.

Exercise 6 When you exercise, your heart rate should increase until it reaches your target heart rate. The table shows target heart rate, $r = f(a)$, as a function of age.

a	20	25	30	35	40	45	50	55	60	65	70
r	150	146	142	139	135	131	127	124	120	116	112

- Find $f(25)$ and $f(50)$.
- Find a value of a for which $f(a) = 135$.

If a function is described by an equation, we simply substitute the given input value into the equation to find the corresponding output, or function value.

Example 7 The function H is defined by $H = f(s) = \frac{\sqrt{s+3}}{s}$. Evaluate the function at the given values.

- $s = 6$
- $s = -1$

Solutions We substitute the given values for s into the equation defining H .

a. $H(6) = \frac{\sqrt{6+3}}{6} = \frac{\sqrt{9}}{6} = \frac{3}{6} = \frac{1}{2}$. Thus, $f(6) = \frac{1}{2}$.

b. $H(-1) = \frac{\sqrt{-1+3}}{-1} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$. Thus, $f(-1) = -\sqrt{2}$. ■

Exercise 7a. Complete the table displaying ordered pairs for the function

$$f(x) = 5 - x^3$$

Evaluate the function to find the corresponding $f(x)$ -value for each value of x .

b. Solve the equation $f(x) = -3$.

x	$f(x)$
-2	
0	
1	
3	

$$f(-2) = 5 - (-2)^3 =$$

$$f(0) = 5 - 0^3 =$$

$$f(1) = 5 - 1^3 =$$

$$f(3) = 5 - 3^3 =$$

Answers to Exercises

1. a. No. Two students with the same grade in English might get different grades in algebra. b. Yes. The number of calories is a multiple of the number of ounces.
 2. a. No. There are two outputs, for example, for $x = 3.5$. b. Yes. Each input has a unique output.
 3. a. d is the input and a is the output. b. About 210 ft
 c. 5 mi, 11 mi, 12 mi, 16.25 mi, 17.3 mi, 17.8 mi d. Max: 300 ft at 0 mi; min: 85 ft at 15 mi
 e. About 170 ft 4. $V = f(r) = \frac{4}{3}\pi r^3$ 5. a. $H = F(t)$ b. On the 15th day of the year, there are 9.7 hours of sunlight.
 6. a. $f(25) = 146$; $f(50) = 127$ b. $a = 40$

7a.

x	-2	0	1	3
$f(x)$	13	5	4	-22

 b. 2

x	-2	0	1	3
$f(x)$	13	5	4	-22

Now go to WeBWork and do the Reading Assignment for this Lesson.

Homework 18

- A** 1. Which of the following tables define the second variable as a function of the first variable? Explain why or why not.

a.

x	t
-1	2
0	9
1	-2
0	-3
-1	5

b.

y	w
0	8
1	12
3	7
5	-3
7	4

c.

x	y
-3	8
-2	3
-1	0
0	-1
1	0
2	3
3	8

d.

s	t
2	5
4	10
6	15
8	20
6	25
4	30
2	35

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e.

r	-4	-2	0	2	4
v	6	6	3	6	8

f.

p	-5	-4	-3	-2	-1
d	-5	-4	-3	-2	-1

2. Which of the following tables define the second variable as a function of the first variable? Explain why or why not.

a.

Pressure (p)	Volume (v)
15	100.0
20	75.0
25	60.0
30	50.0
35	42.8
40	37.5
45	33.3
50	30.0

b.

Frequency (f)	Wavelength (w)
5	60.0
10	30.0
20	15.0
30	10.0
40	7.5
50	6.0
60	5.0
70	4.3

c.

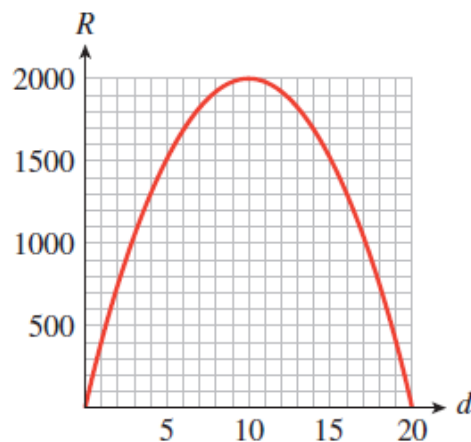
Temperature (T)	Humidity (h)
Jan. 1 34°F	42%
Jan. 2 36°F	44%
Jan. 3 35°F	47%
Jan. 4 29°F	50%
Jan. 5 31°F	52%
Jan. 6 35°F	51%
Jan. 7 34°F	49%

d.

Inflation Rate (I)	Unemployment Rate (U)
1972 5.6%	5.1%
1973 6.2%	4.5%
1974 10.1%	4.9%
1975 9.2%	7.4%
1976 5.8%	6.7%
1977 5.6%	6.8%
1978 6.7%	7.4%

- A** 3. The function described in Problem 2a is called g , so that $v = g(p)$. Find the following.
- a. $g(25)$ b. $g(40)$ c. A value of p so that $g(p) = 75$
4. The function described in Problem 2b is called h , so that $w = h(f)$. Find the following.
- a. $h(20)$ b. $h(60)$ c. A value of f so that $h(f) = 10$

- A** 5. The graph shows the revenue, R , a movie theater collects as a function of the price d it charges for a ticket, so that $R = f(d)$. Write your answer to questions (a) and (b) with function notation.
- a. Estimate the revenue if the theater charges \$12.00 for a ticket.
- b. What should the theater charge for a ticket in order to collect \$1500 in revenue?
- c. For what values of d is $R > 1800$?



6. The graph shows the federal minimum wage, M , over the past five decades, adjusted for inflation to reflect its buying power in 2004 dollars. (Source: www.infoplease.com)



- Is M a function of t ? Support your answer.
 - What is the largest function value on the graph? Write your answer with function notation, and explain what it means in this problem.
 - Give two years in which the minimum wage was worth \$8 in 2004 dollars. Does this fact mean that M is not a function of t ? Why or why not?
- A** 7. The number of compact cars that a large dealership can sell at price p is given by $N = F(p) = \frac{12,000,000}{p}$.
- Evaluate $F(6000)$ and explain what it means.
 - As p increases, does $F(p)$ increase or decrease? Support your answer with calculations.
 - Solve the equation $F(p) = 400$, and explain what it means.
8. The distance, d , in miles that a person can see on a clear day from a height h in feet is given by $d = G(h) = 1.22\sqrt{h}$.
- Evaluate $G(20,320)$ and explain what it means.
 - As h increases, does $G(h)$ increase or decrease? Support your answer with calculations.
 - Estimate the height you need in order to see 100 miles. Write your answer with function notation.

Now go to WeBWork and do the Skills Practice for this Homework.