

# Lesson 11 Intercepts, Solutions, and Factors

In this lesson we study the connection between:

1. the **factors** of  $ax^2 + bx + c$ ,
2. the **solutions** of the quadratic equation  $ax^2 + bx + c = 0$ , and
3. the  **$x$ -intercepts** of the graph of  $y = ax^2 + bx + c$ .

Suppose a baseball player pops up, that is, hits the baseball straight up into the air. The height,  $h$ , of the baseball after  $t$  seconds is given by a formula from physics. This formula takes into account the initial speed of the ball (64 feet per second) and its height when it was hit (4 feet). The formula for the height of the ball is

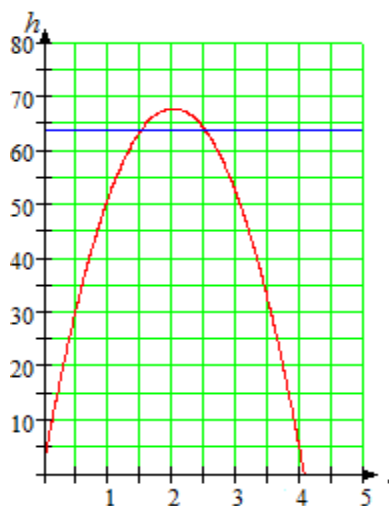
$$h = -16t^2 + 64t + 4$$

We can use a graph to find two times when the baseball was 64 feet high. Is there a way to solve the same problem algebraically?

We are looking for values of  $t$  that produce  $h = 64$  in the height equation. So, if we substitute  $h = 64$  into the formula, we would like to solve the quadratic equation

$$64 = -16t^2 + 64t + 4$$

This equation cannot be solved by extraction of roots, because there are two terms containing the variable  $t$ , and they cannot be combined. To solve this equation, we appeal to a property of our number system called the **zero-factor principle**.



## Zero Factor Principle

Can you multiply two numbers together and obtain a product of zero? Only if one of the two numbers happens to be zero.

### Zero-Factor Principle

The product of two factors equals zero if and only if one or both of the factors equals zero. In symbols,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0$$

**RQ1** Fill in the blanks.

- a. If the sum of two numbers is zero, the numbers must be \_\_\_\_\_.
- b. If a fraction equals zero, the \_\_\_\_\_ must be zero.
- c. If the product of two numbers is zero, one of the numbers must \_\_\_\_\_.
- d. If the divisor in a quotient is zero, the quotient is \_\_\_\_\_.

## 2 Unit 3 Quadratic Models

The principle is true even if the numbers  $a$  and  $b$  are represented by algebraic expressions, such as  $x - 6$  or  $x + 2$ . For example, if

$$(x - 6)(x + 2) = 0$$

then it must be true that either  $x - 6 = 0$  or  $x + 2 = 0$ . Thus, we can use the zero-factor principle to solve equations.

- Example 1a.** Solve the equation  $(x - 6)(x + 2) = 0$   
**b.** Find the  $x$ -intercepts of the graph of  $y = x^2 - 4x - 12$

**Solutions a.** We apply the zero-factor principle to the product  $(x - 6)(x + 2)$ .

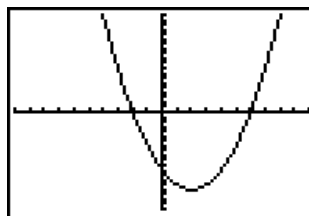
$$\begin{array}{l} (x - 6)(x + 2) = 0 \\ x - 6 = 0 \quad \text{or} \quad x + 2 = 0 \\ x = 6 \quad \text{or} \quad x = -2 \end{array} \quad \begin{array}{l} \text{Set each factor equal to zero.} \\ \text{Solve each equation.} \end{array}$$

There are two solutions, 6 and  $-2$ . You should check that both of these values satisfy the original equation.

- b.** To find the  $x$ -intercepts of the graph, we set  $y = 0$ , and solve the equation

$$0 = x^2 - 4x - 12$$

But this is the same equation we solved in part (a), because  $(x - 6)(x + 2) = x^2 - 4x - 12$ . The solutions of that equation were 6 and  $-2$ , so the  $x$ -intercepts of the graph are 6 and  $-2$ .



$[-10, 10] \times [-20, 10]$

You can see this by graphing the equation on your calculator, as shown in the figure. ■

**Exercise 1** Graph the equation

$$y = (x - 3)(2x + 3)$$

on a calculator, and use your graph to solve the equation  $y = 0$ . (Use  $X_{\min} = -9.4$ ,  $X_{\max} = 9.4$ .) Check your answer with the zero-factor principle.

## Solving Quadratic Equations by Factoring

Before we apply the zero-factor principle, we must first write the quadratic equation so that one side of the equation is zero.

**Example 2** Solve  $3x(x + 1) = 2x + 2$

**Solution** First, we write the equation in standard form.

$$\begin{array}{l} 3x(x + 1) = 2x + 2 \quad \text{Apply the distributive law to the left side.} \\ 3x^2 + 3x = 2x + 2 \quad \text{Subtract } 2x + 2 \text{ from both sides.} \\ 3x^2 + x - 2 = 0 \end{array}$$

Now we factor the left side to obtain

$$\begin{array}{l} (3x - 2)(x + 1) = 0 \quad \text{Apply the zero-factor principle.} \\ 3x - 2 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{Solve each equation.} \\ x = \frac{2}{3} \quad \text{or} \quad x = -1 \end{array}$$

The solutions are  $\frac{2}{3}$  and  $-1$ . ■

**Caution!** When we apply the zero-factor principle, one side of the equation *must* be zero. For example, to solve the equation

$$(x - 2)(x - 4) = 15$$

it is *incorrect* to set each factor equal to 15! (There are many ways that the product of two numbers can equal 15; it is not necessary that one of the numbers be 15.) We must first simplify the left side and write the equation in standard form. (The correct solutions are 7 and  $-1$ ; make sure you can find these solutions.) ■

We summarize the factoring method for solving quadratic equations as follows.

### To Solve a Quadratic Equation by Factoring

1. Write the equation in standard form.
2. Factor the left side of the equation.
3. Apply the zero-factor principle: Set each factor equal to zero.
4. Solve each equation. There are two solutions (which may be equal).

**Exercise 2** Solve by factoring:  $(t - 3)^2 = 3(9 - t)$

Multiply out each side of the equation.

Obtain zero on one side of the equation.

Factor the other side.

Apply the zero-factor principle.

**RQ2** Which technique, extracting roots or factoring, is better-suited to each equation?

a.  $4x^2 - 12x = 0$

b.  $6(4x - 1)^2 = 18$

c.  $(x + 4)^2 = 16x$

d.  $9x^2 - 42 = 0$

We can use factoring to solve the opening problem in this lesson.

■ **Example 3** The height,  $h$ , of a baseball  $t$  seconds after being hit is given by

$$h = -16t^2 + 64t + 4$$

When will the baseball reach a height of 64 feet?

**Solution** We substitute **64** for  $h$  in the formula, and solve for  $t$ .

$$\begin{aligned} 64 &= -16t^2 + 64t + 4 && \text{Write the equation in standard form.} \\ 16t^2 - 64t + 60 &= 0 && \text{Factor 4 from the left side.} \\ 4(4t^2 - 16t + 15) &= 0 && \text{Factor the quadratic expression.} \\ 4(2t - 3)(2t - 5) &= 0 && \text{Set each variable factor equal to zero.} \\ 2t - 3 = 0 & \quad \text{or} \quad 2t - 5 = 0 && \text{Solve each equation.} \\ t = \frac{3}{2} & \quad \text{or} \quad t = \frac{5}{2} \end{aligned}$$

There are two solutions. At  $t = \frac{3}{2}$  seconds, the ball reaches a height of 64 feet on the way up, and at  $t = \frac{5}{2}$  seconds, the ball is 64 feet high on its way down. ■

## 4 Unit 3 Quadratic Models

**Caution!** In Example 3, the factor of 4 does not affect the solutions of the equation at all. You can understand why this is true by looking at some graphs. First, check that the two equations

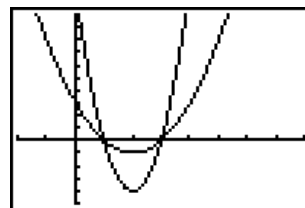
$$x^2 - 4x + 3 = 0$$
$$4(x^2 - 4x + 3) = 0$$

have the same solutions,  $x = 1$  and  $x = 3$ . Then use your graphing calculator to graph the equation

$$Y_1 = X^2 - 4X + 3$$

in the window

$$\begin{aligned} X_{\min} &= -2 & X_{\max} &= 8 \\ Y_{\min} &= -5 & Y_{\max} &= 10 \end{aligned}$$



$[2, 8] \times [-5, 10]$

Notice that when  $y = 0$ ,  $x = 3$  or  $x = 1$ . These two points are the  $x$ -intercepts of the graph. Now on the same window graph

$$Y_2 = 4(X^2 - 4X + 3)$$

This graph has the same  $x$ -values when  $y = 0$ . The factor of 4 makes the graph "skinnier," but does not change the location of the  $x$ -intercepts. ■

**Exercise 3a.** Solve  $4t - t^2 = 0$  by factoring.

**b.** Solve  $20t - 5t^2 = 0$  by factoring.

**c.** Graph  $y = 4t - t^2$  and  $y = 20t - 5t^2$  together in the window

$$\begin{aligned} X_{\min} &= -2 & X_{\max} &= 6 \\ Y_{\min} &= -20 & Y_{\max} &= 25 \end{aligned}$$

and locate the horizontal intercepts of each graph.

**RQ3** Match each equation with its solutions.

- |                                 |                    |
|---------------------------------|--------------------|
| <b>a.</b> $3t(t - 2) = 0$       | <b>I.</b> $-1, 2$  |
| <b>b.</b> $t^2 - t = 2$         | <b>II.</b> $0, 2$  |
| <b>c.</b> $3t^2 = 12$           | <b>III.</b> $3, 2$ |
| <b>d.</b> $t(t - 3) = 2(t - 3)$ | <b>IV.</b> $-2, 2$ |

## Applications

Here is another example of how quadratic equations arise in applications.

■ **Example 4** The size of a rectangular computer monitor screen is taken to be the length of its diagonal. If the length of the screen should be 3 inches greater than its width, what are the dimensions of a 15-inch monitor?

**Solution** We express the two dimensions of the screen in terms of a single variable:

$$\begin{aligned} \text{Width of screen:} & \quad w \\ \text{Length of screen:} & \quad w + 3 \end{aligned}$$

We apply the Pythagorean theorem to write an equation:

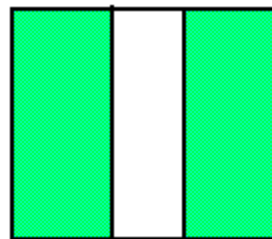
$$w^2 + (w + 3)^2 = 15^2$$

To solve the equation, we begin by simplifying the left side.

$$\begin{array}{ll}
 w^2 + w^2 + 6w + 9 = 225 & \text{Write the equation in standard form.} \\
 2w^2 + 6w - 216 = 0 & \text{Factor 2 from the left side.} \\
 2(w^2 + 3w - 108) = 0 & \text{Factor the quadratic expression.} \\
 2(w - 9)(w + 12) = 0 & \text{Set each factor equal to zero.} \\
 w - 9 = 0 \quad \text{or} \quad w + 12 = 0 & \text{Solve each equation.} \\
 w = 9 \quad \text{or} \quad w = -12 &
 \end{array}$$

Because the width of the screen cannot be a negative number, the width is 9 inches, and the length is  $w + 3 = 12$  inches. ■

**Exercise 4** Francine is designing the layout for a botanical garden. The plan includes a square herb garden, with a path five feet wide through the center of the garden. To include all the species of herbs, the planted area must be 300 square feet. Find the dimensions of the herb garden.



## Solutions of Quadratic Equations

As we have seen in the examples above, the solutions of the quadratic equation

$$a(x - r_1)(x - r_2) = 0$$

are  $r_1$  and  $r_2$ . This is called the **factored form** of the quadratic equation. Thus, if we know the two solutions of a quadratic equation, we can work backwards to reconstruct the equation.

**Example 5** Find a quadratic equation whose solutions are  $\frac{1}{2}$  and  $-3$ .

**Solution** The factored form of the quadratic equation is

$$\begin{array}{l}
 \left(x - \frac{1}{2}\right)[x - (-3)] = 0 \quad \text{Simplify.} \\
 \left(x - \frac{1}{2}\right)(x + 3) = 0
 \end{array}$$

To write the equation in standard form, we multiply the factors together.

$$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

We can also find an equation with integer coefficients if we clear the equation of fractions.

$$\begin{array}{l}
 2\left(x^2 + \frac{5}{2}x - \frac{3}{2}\right) = 2(0) \quad \text{Multiply both sides by 2.} \\
 2x^2 + 5x - 3 = 0
 \end{array}$$

You can check that the solutions of this last equation are in fact  $\frac{1}{2}$  and  $-3$ . Multiplying by 2 does not change the equation's solutions. ■

**Exercise 5** Find a quadratic equation with integer coefficients whose solutions are  $\frac{2}{3}$  and  $-5$ .

## 6 Unit 3 Quadratic Models

A quadratic equation in one variable always has two solutions. In some cases, the solutions may be equal. For example, the equation

$$x^2 - 2x + 1 = 0$$

can be solved by factoring as follows:

$$(x - 1)(x - 1) = 0 \quad \text{Apply the zero-factor principle.}$$
$$x - 1 = 0 \quad \text{or} \quad x - 1 = 0$$

Both of these equations have solution 1. We say that 1 is a solution of **multiplicity** two, meaning that it occurs twice as a solution of the quadratic equation.

**RQ4** True or false.

- We find the  $x$ -intercepts of a graph by setting  $x = 0$ .
- If  $z$  is a solution of a quadratic equation, then  $(x - z)$  is a factor of the left side in standard form.
- We can factor a constant from both sides of a quadratic equation without changing its solutions.
- To solve a quadratic equation by factoring, we should factor each side of the equation.

### Answers to Exercises

- $x = 3, x = \frac{-3}{2}$
- $t = 6, t = -3$
- a.  $t = 0, t = 4$     b.  $t = 0, t = 4$
- $t = 0, t = 4$
- 20 ft by 20 ft
- $3x^2 + 13x - 10 = 0$

**Now go to WeBWork and do the Reading Assignment for this Lesson.**

## Homework 11

- Graph the function in the **ZInteger** window, and locate the  $x$ -intercepts of the graph. Use the  $x$ -intercepts to write the quadratic expression in factored form.

**A** 1.  $y = 0.1(x^2 - 3x - 270)$                       2.  $y = -0.06(x^2 - 22x - 504)$

3. Delbert stands at the top of a 300-foot cliff and throws his algebra book directly upward with a velocity of 20 feet per second. The height of his book above the ground  $t$  seconds later is given by the equation

$$h = -16t^2 + 20t + 300$$

where  $h$  is in feet.

- Use your calculator to make a table of values for the height equation, with increments of 0.5 second.
- Graph the height equation on your calculator. Use your table of values to help you choose appropriate **WINDOW** settings.

- c. What is the highest altitude Delbert's book reaches? When does it reach that height? Use the **TRACE** feature to find approximate answers first. Then use the **Table** feature to improve your estimate.
- d. When does Delbert's book pass him on its way down? (Delbert is standing at a height of 300 feet.) Use the **intersect** command.
- e. Write and solve an equation to answer the question: How long will it take Delbert's book to hit the ground at the bottom of the cliff?

4. The annual increase  $I$  in the deer population in a national park is given by the formula

$$I = 1.2x - 0.0002x^2$$

where  $x$  is the size of the population that year.

- a. Make a table of values for  $I$  for  $0 \leq x \leq 7000$ . Use increments of 500 in  $x$ .
- b. How much will a population of 2000 deer increase? A population of 5000 deer? A population of 7000 deer?
- c. Use your calculator to graph the annual increase  $I$  versus the size of the population,  $x$ , for  $0 \leq x \leq 7000$ .
- d. What do the  $x$ -intercepts tell us about the deer population?
- e. Estimate the population size that results in the largest annual increase. What is that increase?

- A** 5. One end of a ladder is 10 feet from the base of a wall, and the other end reaches a window in the wall. The ladder is 2 feet longer than the height of the window.
- a. Choose a variable for the height of the window. Make a sketch of the situation described, and label the sides of a right triangle.
  - b. Write a quadratic equation about the height of the window.
  - c. Solve your equation to find the height of the window.

6. Irene would like to enclose two adjacent chicken coops of equal size against the henhouse wall. She has 66 feet of chicken wire fencing, and she would like the total area of the two coops to be 360 square feet. What should the dimensions of the chicken coops be?



We'll use three methods to solve this problem: a table of values, a graph, and an algebraic equation.

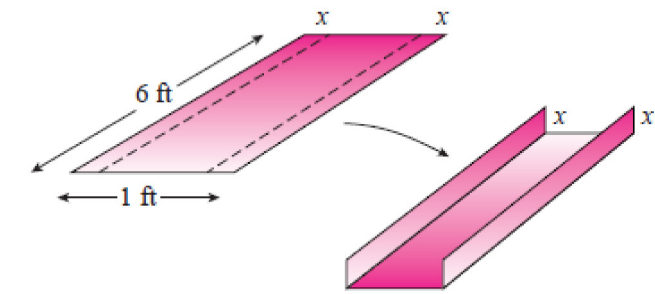
- a. Make a table by hand that shows the areas of coops of various widths, as shown at right. Continue the table until you find a pair of chicken coops whose area is 360 square feet. (Be careful computing the length of the coops: look at the diagram above.)

Width	Length	Area
2	60	120
4	54	216
⋮	⋮	⋮

- b. Write an expression for the length of the two coops if their width is  $x$ . Then write an expression for the area  $A$  of the coops if their width is  $x$ . Graph the equation for  $A$ , and use the graph to find the pair of coops whose area is 360 square feet. (Is there more than one solution?)
- c. Write an equation for the area  $A$  of the two coops in terms of their width,  $x$ . Solve your equation algebraically for  $A = 360$ .

## 8 Unit 3 Quadratic Models

7. A length of rain gutter is made from a piece of aluminum 6 feet long and 1 foot wide.
- If a strip of width  $x$  is turned up along each long edge, write expressions for the length, width and height of the gutter. Then write an expression for the volume  $V$  of the gutter in terms of  $x$ .
  - Use your calculator to make a table of values showing the volumes of various rain gutters formed by turning up edges of 0.1 foot, 0.2 foot, and so on.
  - Graph your expression for the volume. What happens to  $V$  as  $x$  increases?
  - Use your table or your graph to discover how much metal should be turned up along each long edge so that the gutter has a capacity of  $\frac{3}{4}$  cubic foot of rainwater.
  - Write and solve a quadratic equation to answer part (d).



**Now go to WeBWork and do the Skills Practice for this Homework.**