

# Work Toward Verification of a Geometric Construction

## The Regular Pentagon

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August 29, 2007

### Given:

Steps for the geometric construction of a regular pentagon:

1. Construct a circle – this circle, with radius  $R$ , will circumscribe the pentagon and will be called the *circumcircle*
2. Construct two perpendicular diameters - oriented such that they will become the axes of a Cartesian coordinate system
3. Bisect a radius -the positive x-axis
4. Construct a circle with the midpoint of the radius as the center and with a radius of  $\frac{R}{2}$  – the *small circle*
5. Draw a line from the intersection of the *circumcircle* with the negative y-axis thru the center of the *small circle*
6. Construct a circle with center as the intersection of the *circumcircle* with the negative y-axis and a radius determined by the furthestmost point of intersection between the line –last step– and the *small circle* – this will be called the *big circle*
7. The points of intersection between the *big circle* and the *circumcircle* and the intersection of the positive y-axis with the *circumcircle* form three of the vertices for the pentagon
8. The remaining vertices can be obtained by marking off similar chords on the *circumcircle* – see fig. 1

**Find:** Obtain the  $(x, y)$  coordinates of the vertices and confirm that it is a regular pentagon by showing that sides are the same length and the angles are equal to  $108^\circ$

### Plan:

Use diameters as x-y coordinate axes. Get equations for 3 circles; namely the *circumcircle*, the *small circle* and the *big circle*. By finding the points of intersection between the *big circle* and *circumcircle* we have two (2) of the vertices. Another vertex, as seen from the construction, is  $(0, R)$ . Using these three we can get lengths of two sides and the angle between them.

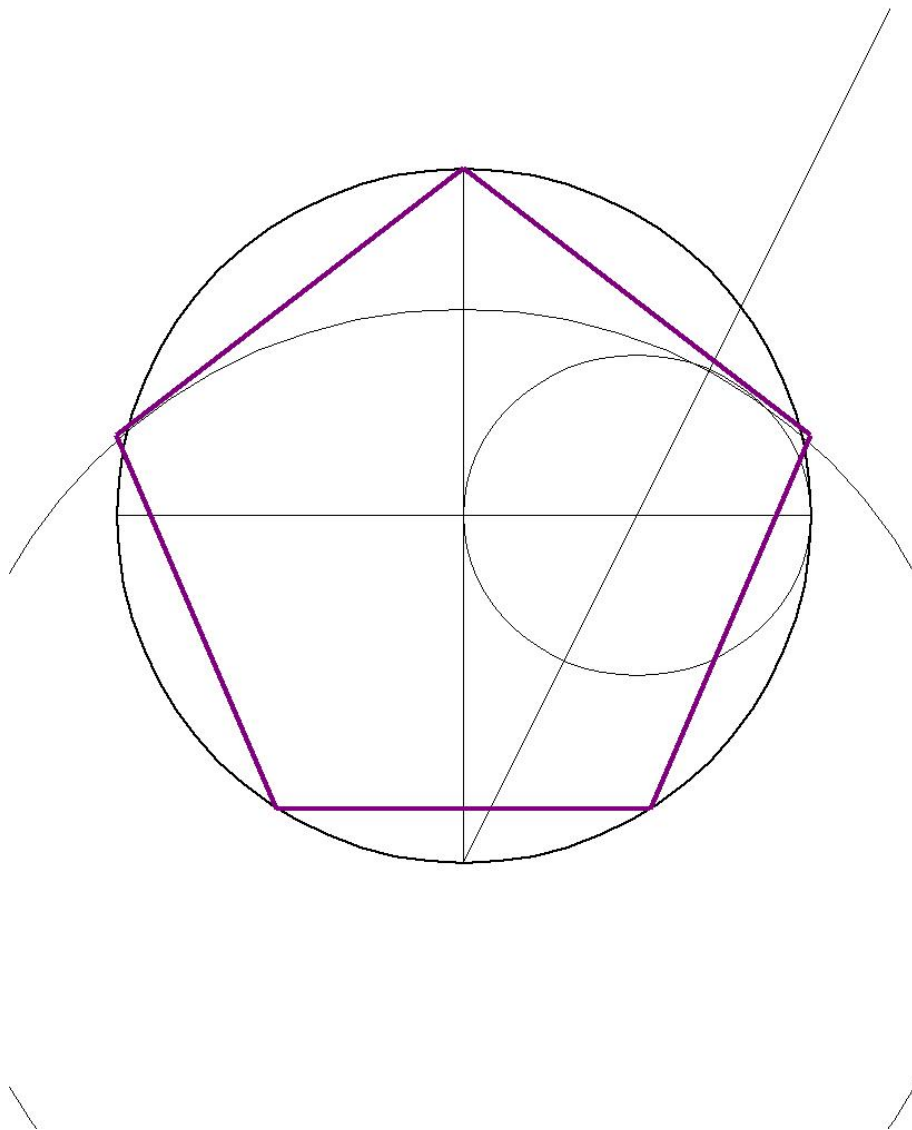


Figure 1: Sketch of Pentagon Construction

**Calculations:**

Equation of the *circumcircle*:

$$x^2 + y^2 = R^2 \quad (1)$$

Equation of the *small* circle:

$$\left(x - \frac{R}{2}\right)^2 + y^2 = \left(\frac{R}{2}\right)^2 \quad (2)$$

Equation of the line:

$$y = 2x - R \quad (3)$$

Now we find the point of intersection between the *small* circle and the line. Substituting:

$$\begin{aligned} \left(x - \frac{R}{2}\right)^2 + (2x - R)^2 &= \left(\frac{R}{2}\right)^2 \\ x^2 - Rx + \frac{R^2}{4} + 4x^2 - 4Rx + R^2 &= \frac{R^2}{4} \\ 5x^2 - 5Rx + R^2 &= 0 \end{aligned}$$

Solving this quadratic equation for x by completing the square:

$$\begin{aligned} x^2 - Rx &= -\frac{R^2}{5} \\ \left(x - \frac{R}{2}\right)^2 &= \frac{R^2}{4} - \frac{R^2}{5} \\ x - \frac{R}{2} &= \pm \sqrt{\frac{R^2}{20}} \\ x &= R \left(\frac{1}{2} \pm \frac{1}{2\sqrt{5}}\right) \end{aligned}$$

From the diagram we want:

$$x = R \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)$$

So, since  $y = 2x - R$ ,

$$\begin{aligned} y &= 2R \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right) - R \\ y &= R + \frac{R\sqrt{5}}{5} - R \\ y &= \frac{R\sqrt{5}}{5} \end{aligned}$$

The distance from  $(0, -R)$  to  $\left(R\left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right), \frac{R\sqrt{5}}{5}\right)$  is the radius of the *big* circle or:

$$\begin{aligned}
 R_B^2 &= \left(R\left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right) - 0\right)^2 + \left(\frac{R\sqrt{5}}{5} - (-R)\right)^2 \\
 &= R^2 \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)^2 + R^2 \cdot \frac{1}{5} + 2R^2 \frac{\sqrt{5}}{5} + R^2 \\
 &= R^2 \left[\frac{1}{4} + \frac{\sqrt{5}}{10} + \frac{1}{20} + \frac{1}{5} + \frac{2\sqrt{5}}{5} + 1\right] \\
 R_B^2 &= R^2 \left[\frac{3}{2} + \frac{\sqrt{5}}{2}\right]
 \end{aligned}$$

Since the center of the *big* circle is  $(0, -R)$ , the equation of the *big* circle is:

$$x^2 + (y + R)^2 = R^2 \left[\frac{3}{2} + \frac{\sqrt{5}}{2}\right] \quad (4)$$

Now we are after the intersection between the *big* circle and the *circumcircle*. By solving for  $x^2$  in equation 1 and substituting into equation 4, we can find the coordinates of two of the vertices.

$$\begin{aligned}
 (R^2 - y^2) + (y + R)^2 &= R^2 \left[\frac{3}{2} + \frac{\sqrt{5}}{2}\right] \\
 R^2 - y^2 + y^2 + 2Ry + R^2 &= R^2 \left[\frac{3}{2} + \frac{\sqrt{5}}{2}\right] \\
 2Ry + 2R^2 &= R^2 \left[\frac{3}{2} + \frac{\sqrt{5}}{2}\right] \\
 y + R &= R \left[\frac{3}{4} + \frac{\sqrt{5}}{4}\right] \\
 y &= R \left[\frac{3}{4} + \frac{\sqrt{5}}{4} - 1\right] \\
 y &= R \left[\frac{\sqrt{5} - 1}{4}\right]
 \end{aligned}$$

Plugging this value for y into equation 1 will give us the x coordinates.

$$\begin{aligned}
 x^2 + \left(R \left[\frac{\sqrt{5} - 1}{4}\right]\right)^2 &= R^2 \\
 x^2 &= R^2 \left[1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2\right]
 \end{aligned}$$

$$\begin{aligned}
&= R^2 \left[ 1 - \frac{5 - 2\sqrt{5} + 1}{16} \right] \\
&= R^2 \left[ 1 - \frac{2(3 - \sqrt{5})}{16} \right] \\
&= R^2 \left[ \frac{8}{8} - \frac{(3 - \sqrt{5})}{8} \right] \\
&= R^2 \left[ \frac{5 + \sqrt{5}}{8} \right] \\
x &= \pm R \left[ \frac{5 + \sqrt{5}}{8} \right]^{\frac{1}{2}}
\end{aligned}$$

And so we now have the coordinates of two vertices and the vertex between them, namely:

$$\begin{aligned}
&\left( \pm R \left[ \frac{5 + \sqrt{5}}{8} \right]^{\frac{1}{2}}, R \left[ \frac{\sqrt{5} - 1}{4} \right] \right) \\
&(0, R)
\end{aligned}$$

Now we want to show that the distances of the two sides formed by these three vertices are equal in length. It might also be fun to find the actual length of a side of the pentagon in terms of the *circumcircle* radius. Then our final task will be to find the angle between the two sides. – see fig. 2 Using the distance formula and symmetry, we can find the lengths of the sides.

$$\begin{aligned}
S_1^2 = S_2^2 &= \left( 0 - \left( \pm R \left[ \frac{5 + \sqrt{5}}{8} \right]^{\frac{1}{2}} \right) \right)^2 + \left( R - R \left[ \frac{\sqrt{5} - 1}{4} \right] \right)^2 \\
&= R^2 \left[ \frac{5 + \sqrt{5}}{8} \right] + R^2 \left( 1 - \left[ \frac{\sqrt{5} - 1}{4} \right] \right)^2 \\
&= R^2 \left[ \frac{5}{8} + \frac{\sqrt{5}}{8} + 1 - 2 \left[ \frac{\sqrt{5} - 1}{4} \right] + \left[ \frac{\sqrt{5} - 1}{4} \right]^2 \right] \\
&= R^2 \left[ \frac{5}{8} + \frac{\sqrt{5}}{8} + 1 - \frac{\sqrt{5}}{2} + \frac{1}{2} + \frac{(5 - 2\sqrt{5} + 1)}{16} \right] \\
&= R^2 \left[ \frac{10}{16} + \frac{16}{16} + \frac{8}{16} + \frac{6}{16} + \frac{2\sqrt{5}}{16} - \frac{8\sqrt{5}}{16} - \frac{2\sqrt{5}}{16} \right] \\
&= R^2 \left[ \frac{5 - \sqrt{5}}{2} \right] \\
S_1^2 = S_2^2 &= R^2 \left[ \frac{5 - \sqrt{5}}{2} \right]
\end{aligned}$$

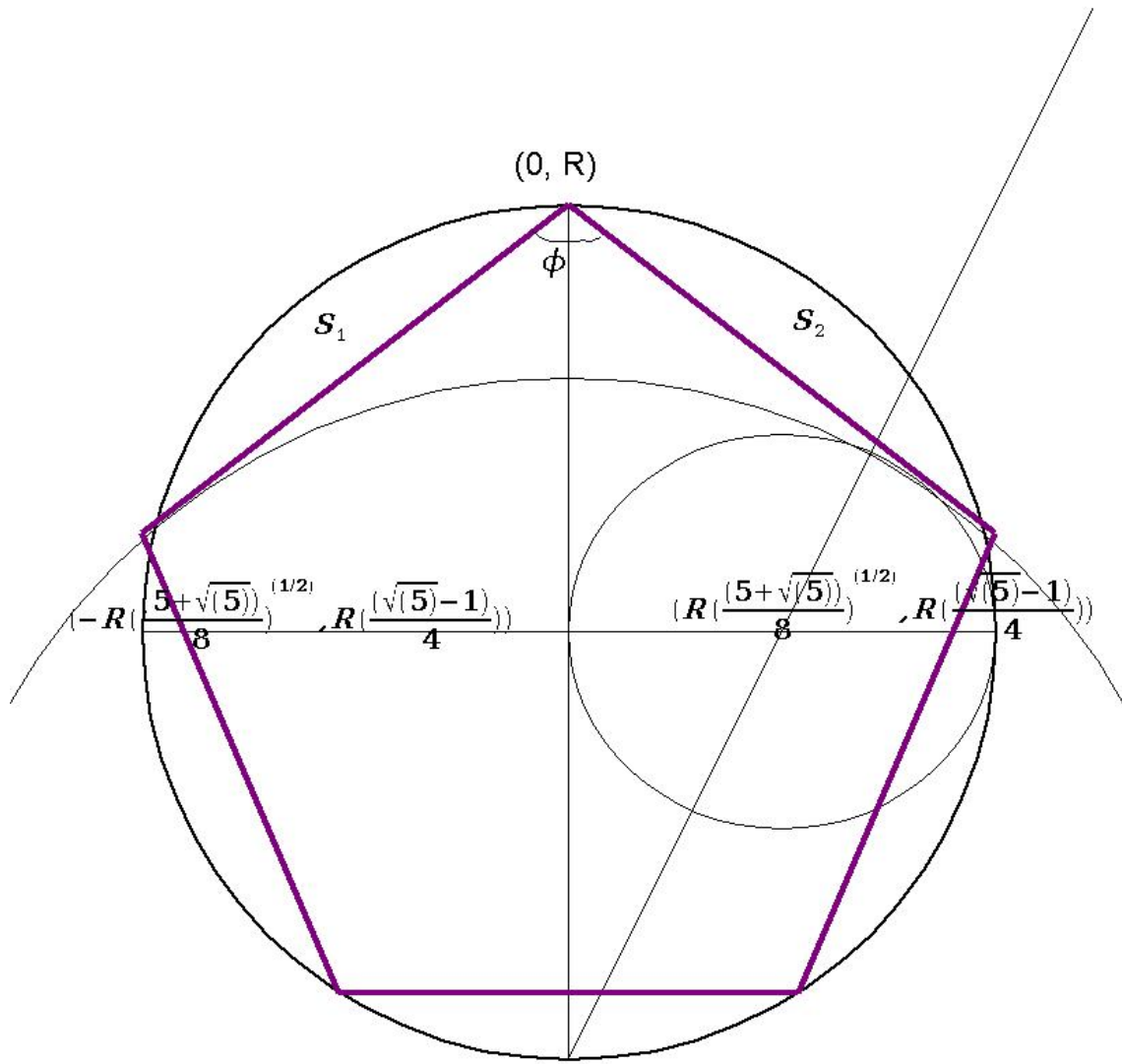


Figure 2: Sketch of Pentagon with Vertices Labeled

For some background concerning determining the angle between two lines, see figure 3. By finding the slopes of the sides of the pentagon  $S_1$  and  $S_2$  and plugging into equation 5 we will be able to check to see if the angle is  $108^\circ$  as it should be for a regular pentagon. The slope of  $S_1$  is:

$$\begin{aligned}
 m_1 &= \frac{\Delta y}{\Delta x} \\
 m_1 &= \frac{R - R \left[ \frac{\sqrt{5}-1}{4} \right]}{0 - \left( -R \left[ \frac{5+\sqrt{5}}{8} \right]^{\frac{1}{2}} \right)} \\
 &= \frac{R \left[ 1 - \frac{\sqrt{5}}{4} + \frac{1}{4} \right]}{R \left[ \frac{5+\sqrt{5}}{8} \right]^{\frac{1}{2}}} \\
 m_1 &= \frac{\left( \frac{5-\sqrt{5}}{4} \right)}{\left( \frac{5+\sqrt{5}}{8} \right)^{\frac{1}{2}}}
 \end{aligned}$$

Now  $m_1 = -m_2$  so,

$$\tan \phi = \frac{m_2 - m_1}{1 + m_1 \cdot m_2} \quad (5)$$

$$\tan \phi = \frac{-m_1 - m_1}{1 - m_1^2}$$

$$= \frac{-2m_1}{1 - m_1^2}$$

$$\phi = \arctan \frac{-2m_1}{1 - m_1^2}$$

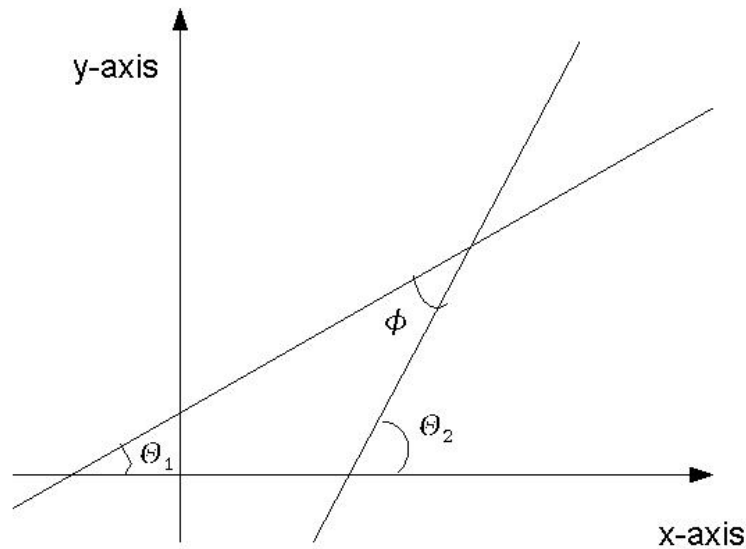
By simply entering the value for the slope into a calculator, we get a decimal approximation.

$$m_1 \approx 0.7265425283$$

$$\phi \approx \arctan(-3.077683542)$$

$$\approx -72^\circ = 108^\circ$$

**Conclusion:** While this is no formal proof, by any stretch of the imagination, I believe we have made a reasonable argument that the geometric construction technique for a regular pentagon can be verified. I may pursue this further, but for now leave it for your enjoyment. Best regards!



From a simple geometry theorem for angles in triangles

$$\phi + \theta_1 = \theta_2 \quad \text{therefore} \quad \phi = \theta_2 - \theta_1 \quad \text{and}$$

$$\tan(\phi) = \tan(\theta_2 - \theta_1) \quad \text{then by using a trig. identity,}$$

$$\tan(\phi) = \frac{(\tan(\theta_2) - \tan(\theta_1))}{(1 + \tan(\theta_1)\tan(\theta_2))}$$

finally using the definition of slope in combination with right triangle trigonometry and the convention of measuring angles in a counter-clockwise direction ...

$$m_2 = \tan(\theta_2) \quad \text{slope of the "terminal" side}$$

$$m_1 = \tan(\theta_1) \quad \text{slope of the "initial" side}$$

$$\tan(\phi) = \frac{(m_2 - m_1)}{(1 + m_1 \cdot m_2)}$$

Figure 3: Deriving Formula for the Angle Between Two Lines