

Using Students' Work as a Lens on



Algebraic Thinking

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USING STUDENTS' WORK IN TEACHERS' PROFESSIONAL development programs has become popular for two reasons:

1. Teachers must consider teaching and learning issues jointly. By putting its focus on learners, the standards movement in mathematics education has increased the demand on professional development to help teachers understand issues of both teaching and learning and to help them interweave their knowledge of learning with their knowledge of teaching. In analyzing students' work together, teachers have numerous opportunities to focus on learning issues and how those concerns link with pedagogy.

2. Teachers must examine their own beliefs about the purpose, methods, and goals of mathematics education. Making inferences about students' mathematical abilities from their written work requires making decisions using information from outside sources and individual teacher's knowledge, beliefs, assumptions, and mind-sets—all of which affect the interpretation of the information.

We must distinguish different purposes for looking at students' work (Driscoll and Bryant 1998). For example, the purpose of understanding students' thinking differs from the purpose of evaluating students' achievement, which in turn differs from the purpose of improving instruction. The main focus of this article is on how teachers can use students' work to understand students' thinking. A secondary focus is how such knowledge can be used to improve instruction. This article does not discuss issues related to evaluating students' achievement.

Understanding Students' Algebraic Thinking

IN THE LINKED LEARNING IN MATHEMATICS Project (LLMP), a program for middle school and high school teachers in Milwaukee, students' work serves as one tool to help teachers gain a deeper understanding of algebraic thinking. LLMP follows from two preceding projects: Leadership for Urban Mathematics Reform (LUMR) and Assessment Communities of Teachers (ACT). All three were funded by the National Science Foundation as teacher-enhancement projects. In this article, we describe how we have learned to use students' work to stay consistent with the purpose of understanding students' algebraic thinking. A more comprehensive account of our approach to algebraic thinking appears in *Fostering Algebraic Thinking: A Guide for Teachers in Grades 6 through 10* (Driscoll 1999).

We do not attempt to establish a comprehensive definition of algebraic thinking. Instead, our work-

ing definition encompasses several patterns of thinking noted in productive algebraic thinkers. The capacity for algebraic thinking includes the ability to think about *functions* and how they work and about the impact of a system's *structure* on calculations. These two aspects of algebraic thinking are facilitated by certain habits of mind:

- **Doing and undoing.** Effective algebraic thinking sometimes involves reversibility, that is, the ability to undo mathematical processes, as well as to do them. This ability means that the mathematician not only uses a process to get to a goal but also understands the process well enough to work backward from the answer to the starting point. For example, in a traditional algebra setting, algebraic thinkers are able not only to solve an equation, such as $9x^2 - 16 = 0$, but also to answer the question "What is an equation with solutions of $4/3$ and $-4/3$?"
- **Building rules to represent functions.** Essential in algebraic thinking is the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined function rules. For example, the following is a function rule that is based on computation: "Choose an input number, multiply it by 4, and subtract 3." This habit of mind is a natural complement to doing and undoing, in that the ability to understand how a function rule works in reverse generally makes the process more accessible and useful.
- **Abstracting from computation.** This ability enables mathematicians to think about computations independently of particular numbers used. One evident characteristic of algebra is that it is abstract. Thinking algebraically involves being able to think about computations apart from the particular numbers to which they are tied in arithmetic, that is, abstracting system regularities from computation. For example, students abstract from computation when they regroup numbers into pairs that equal 101 to make the following computation simpler: "Compute: $1 + 2 + 3 + \dots + 100$."

A Lens on Algebraic Thinking

THE FOLLOWING TWO CONDITIONS ARE REQUIRED for students' work to become an effective professional development lens on algebraic thinking:

1. Criteria for choosing activities and selections of students' work, and
2. A process for analyzing and discussing students' work.

Crossing the River

Eight adults and two children need to cross a river. A small boat is available that can hold one adult, or one or two children. (So, the three possibilities are: 1 adult in the boat; 1 child in the boat; or 2 children in the boat.) Everyone can row the boat. How many one-way trips does it take for them all to cross the river? Can you describe how to work it out for 2 children and any number of adults? How does your rule work out for 100 adults?

What happens to the rule if there are different numbers of children? For example: 8 adults and 3 children? 8 adults and 4 children? Write a rule for finding the number of trips needed for A adults and C children.

One group of adults and children took 27 trips. How many adults and children were in the group? Is there more than one solution?

Fig. 1 The Crossing the River problem

The following sections describe how LLMP has dealt with these conditions.

Criteria for choosing activities and students' work

LLMP looks for activities that encourage honing one or more of the habits of mind described previously. Because the project serves teachers from sixth through tenth grade, we also look for activities that allow for multiple approaches. In addition, the activities should permit extensions to related kinds of algebraic thinking. One such activity is Crossing the River (see fig. 1).

The selection of students' work should be representative of the possible approaches to the activity. For a professional development session, the selection should be limited to four to six pieces but should include pieces that represent several different approaches to the problem and a few pieces in which the students' thinking is not absolutely clear. The responses of two students, Barbara and Denzel, from among those used for one of our sessions are shown in figures 2 and 3. Note that in answering the first question, Barbara uses both Spanish and English, saying in effect, "17 go, 16 return = 33 one-way trips."

Process for analyzing students' work

Using a model called *structured exploration* from the LUMR project, LLMP teachers first immersed themselves in the investigation. Working in small

CROSSING THE RIVER

1. Eight Adults and two children need to cross a river. A small boat is available that can hold one adult, or one or two children. Everyone can row the boat. How many one-way trips does it take for them to all cross the river?

17 in, 16 volver = 33 one way trips

2. What if there were

- *6* adults and two children? *13, 12 = 25 one way trips*
- *15* adults and two children? *31, 30 = 61 one way trips*
- *3* adults and two children? *7, 6 = 13 one way trips*

3. Can you describe, in words, how to figure out the answer to this problem if the group of people to cross the river includes two children and any number of adults? How does your rule work out for 100 adults? *you take the adults and multiply by 2 = 100 x 2 = 200 you add 1 for the first trip = 200 then you add 200 + 201 = 401*

4. Can you write the rule for A number of adults and 2 children?

2a + 2a + 1

5. What happens to the rule you wrote if we change the number of children? For example:

- *8* adults and 3 children? *2a + 1 + 2a + 2 = 18 in, 17 volver = 35 one way trips*
- *2a + 3 + 2a + 4 = 8 in, 7 volver = 15 one way trips*
- Any number A of adults and 11 children?

2a + 9 + 2a + 10

6. One group of adults and children took 27 trips to cross the river. How many adults and how many children were in the group? Is there more than one solution?

Start Finish

2 kids	→	1 kid
←	1 kid	
1 adult	→	
←	1 kid	
2 kids	→	
←	1 kid	
1 adult	→	
←	1 kid	
2 kids	→	
←	1 kid	
1 adult	→	
←	1 kid	
2 kids	→	
←	1 kid	
1 adult	→	
←	1 kid	

Fig. 2 Barbara's work

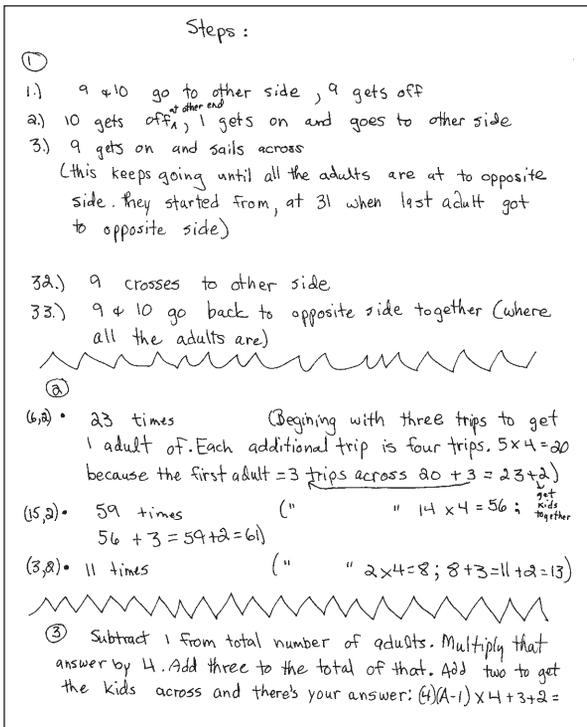


Fig. 3 Denzel's work

groups on the Crossing the River problem, they developed solution strategies, which they then presented to the full group. In the second step of the structured-exploration process, teachers worked again in small groups to analyze students' work. They compared the similarities and differences in algebraic thinking among the students, using the guidelines for analyzing algebraic thinking (see **fig. 4**) to help them focus on the algebraic habit of mind, "building rules to represent functions."

Sample Analysis

THE FOLLOWING ANALYSIS CAPTURES THE teachers' discussions of Barbara's and Denzel's work, using each of the parameters in the guidelines.

Indicator A in the guidelines is "systematically searches for a rule." When taken together, Barbara's responses to parts 1 and 2 of the problem indicate that she is systematically searching for a rule. In the diagram that Barbara drew in response to part 1, right and left arrows represent the direction of the trips needed to get the first 4 adults across in the 8-adult, 2-child situation. Note that the second part of Barbara's diagram, which is not shown, is identical except for a final additional trip in which the "2 kids" move from left to right. Barbara's words, "17 *ir*" (to go) "16 *volver*" (to return), indicate that she has mentally organized the trips into two groups, those in one direction and those in the other. In part 2, Barbara has drawn circles

around the numbers 6, 15, and 3. This tactic probably indicates that she is searching for a rule to connect the varying number of adults with the corresponding numbers of "going" and "returning" trips.

Denzel's combined work on parts 1 and 2 shows that he is paying attention to a repeated group of 4 trips. We can infer from Denzel's statement in part 1, "this keeps going . . .," that he is searching for a rule based on repetitions of same-sized groups of trips. This approach is in contrast with Barbara's, in which she searched for a rule based on two groups ("to go" and "to return") with varying sizes. The composition of Denzel's groups, however, is difficult to ascertain from part 1 alone. Although Denzel's answer to part 1, 33 trips, is correct, the enumeration of the trips is somewhat flawed. Step 2

Overarching question: What do these students know about building rules to represent functions? In looking for evidence, you may notice some of the following:

A. Systematically searches for a rule. Pays attention to how things are changing, perhaps to what gets repeated. (Ask yourself: what is the student paying attention to?)

E.g., "I'll draw out all the steps, then count how many trips in each direction."

E.g., "I've been comparing all the numbers I generated that give the number of trips needed for two adults, and they are all odd."

B. Attempts to describe a rule.

E.g., "You get one adult over, then repeat the steps five times."

E.g., "To find out how many trips are needed for one more adult, add 4." (This describes a NOW-NEXT rule.)

C. Conjectures a generalized rule. (Ask yourself: how does the student think things are working here?)

E.g., "With 2 children, it doesn't matter how many adults you have."

E.g., "From the pattern in the table it looks like you can find the number of trips by multiplying the number of adults by 4 and adding 1."

D. Represents this rule in other forms:

Table ___ Graph ___ Equation ___ Expression ___
E.g., $4A + 1 = T$ E.g., $2a + 2a + 1$

E. Links are made among the different representations to show how they connect. (Ask yourself: does the student seem to see connections between the different representations used?)

E.g., "The 4 in the formula means if the number of adults gets bigger by 1, the number of trips gets bigger by 4."

Fig. 4 Guidelines for analyzing student work

actually involves 2 trips; child 10 gets off after going to the “other end” and adult “1 gets on and goes to the other side.” Furthermore, Denzel does not reveal until part 2 that he is thinking of an initial group of 3 trips to get the first adult across, followed by 7 groups of 4 trips each to get the next 7 adults across, and ending with 2 final trips to get both children across.

Indicator B in the guidelines is “Attempts to describe a rule,” which Barbara does not do until part 3. Unlike Denzel’s first attempt, Barbara’s response has the feel of a generalized conjecture (see indicator C).

In part 2, we see Denzel’s first attempt to describe the rule verbally. At first glance, his answers of 23, 59, and 11 appear incorrect. However, a careful reading of his verbal description reveals that these answers are meant to be the number of trips needed to get 6, 15, and 3 adults across, but not both children. In each case, Denzel parenthetically adds 2 trips to “get kids to the other side,” thus obtaining the correct answers of 25, 61, and 13 trips.

Indicator C, “Conjectures a generalized rule,” is shown in Barbara’s work on part 3, in which she verbally conjectures a rule for finding the number of trips needed in the 100-adult, 2-child situation. In fact, her description has the feel of a generalized conjecture, which she illustrates using the 100-adult example. We cannot tell whether Barbara conjectures the “multiply by 2” rule from her number pattern in part 2 or from the diagram of the situation itself in part 1, which shows two right arrows and two left arrows for each adult.

Part 3 of Denzel’s work shows that he has generalized the specific descriptions of the rule given in part 2. His rule is given in both verbal and symbolic forms. Note, too, that Denzel consolidated all the computations into a single string in part 3, in contrast with the operation fragments that he wrote in part 2 (e.g., $56 + 3 = 59 + 2 = 61$).

Indicator D in the guidelines is “Represents this rule in other forms.” Barbara’s work on part 4 shows that she uses the letter a consistently to represent the number of adults. Her expression $2a + 2a + 1$ is an algebraic representation of the verbal rule that she wrote in part 3. Part 5 shows how Barbara modifies her $2a + 2a + 1$ rule to fit situations with different numbers of children. This work implies that Barbara has conjectured a rule that extends her first rule. In particular, the regular form of the expressions in part 5 shows that Barbara is able to incorporate both types of repetition, trips for adults and children, into her rule, effectively handling a two-variable situation with just one variable. In other words, the expressions in part 5 indicate Barbara’s conjecture that for each child beyond the first two, the rule must be modified by adding 1 to $2a$ and 1 to

$2a + 1$. We cannot be sure whether Barbara realizes that the extra 1’s represent 1 extra trip “to go” and 1 extra trip “to return” or whether they are just a result of a numeric pattern that she analyzed. Finally, although she may recognize a pattern for creating the different rules, Barbara does not seem able to represent the different rules as one single rule with two variables, for example, $2a + (k - 2) + 2a + (k - 1)$.

Part 3 of Denzel’s work shows that he can write the rule in both verbal and algebraic forms. From his verbal description, we can infer that Denzel probably uses $A - 1$ to represent “subtract 1 from [the] total number of adults.” That is, for Denzel, the expression $A - 1$ probably means “perform the operation of subtraction and get an answer” instead of “the number of adults who still need to cross the river after the first has done so.” Note that the “(4)” at the beginning of the string “(4) $(A - 1) \times 4 + 3 + 2$ ” apparently indicates that Denzel is answering part 4 of the activity.

Indicator E directs teachers to look for connections among the different representations. Barbara’s work on part 4 indicates that she has probably written her verbal rule as an algebraic expression, but she does not explicitly describe the connection between the two. In addition, we cannot tell whether Barbara’s conception of the repetitions in the problem leads to the algebraic expression $2a + 2a + 1$ or whether the expression is simply the result of the number pattern that she noticed in part 2.

Denzel’s work on part 2 shows that he has linked the arithmetic computations with the situation itself. His conceptualization eventually leads to the expression $(A - 1) \times 4 + 3 + 2$, which is algebraically, but not conceptually, equivalent to Barbara’s. Parts 3 and 4 show that Denzel has linked the verbal rule with the symbolic form. His consolidation of the multiple steps in part 2 into a single string shows that he has linked the arithmetic computations with the generalized rule.

When asked, “What does this student seem to know about building rules to represent functions?” the teachers concluded that Barbara seems to be able to explicitly build rules that represent patterns and to use algebraic symbolism to represent the rules. Whether Barbara connected the numeric pattern in part 2 with the diagram of the situation itself is unclear. She may not explicitly realize, for example, that the underlying reason for multiplying the number of adults by 2 is that two right-pointing arrows are drawn for each adult. Barbara may simply notice that the numbers in part 2, that is, 12, 30, and 6, are twice the circled numbers of adults. Future interactions with Barbara should encourage her to connect numeric patterns and rules with actual situations.

Unlike Barbara, Denzel explicitly describes the connection between a numeric pattern for comput-

ing the number of trips and the situation itself. Denzel apparently interprets $A - 1$ as a subtraction problem that produces a number rather than a variable expression representing a number of adults. This interpretation shows that Denzel is able to link arithmetic computation with generalized rules; however, it may reveal a weakness in his understanding of algebraic symbolism. Future interactions with Denzel should encourage further and more powerful uses of algebraic symbols to represent quantities in realistic situations.

Improving Instruction

THE ULTIMATE GOAL OF PROFESSIONAL DEVELOPMENT is to improve instruction. Once teachers recognize the levels of algebraic thinking in their students, they need to plan how to help their students raise those levels. Planning appropriate algebra-based activities is not enough; appropriate implementation is also required. In particular, using effective questioning techniques that encourage the development of algebraic habits of mind is important. We worked with the LLMP teachers to identify the types of questions that they might use to help students of varying abilities develop their thinking further. **Figure 5** lists some of the questions that teachers can ask to help students sharpen their algebraic thinking. The LLMP teachers learned to pose such questions, selecting them by analyzing their students' existing algebraic capacity and identifying realistic expectations for growth.

Conclusion

STUDENTS' WORK CAN BE A CATALYST FOR EFFECTIVE professional development for teachers, but its use should be consistent with defined purpose. In LLMP, our purpose is to help teachers gain a deeper understanding of algebraic thinking. Rubrics and other tools for evaluating achievement are not appropriate for that purpose; instead, we fashioned our own set of criteria on the basis of a framework of algebraic habits of mind.

Our experience suggests that students' work can be the basis for designing similar professional development structures to focus on other kinds of mathematical thinking. For a particular content area, such as geometry, measurement, or statistics, two questions need to be asked: (1) What indicators characterize productive thinking in this area? and (2) What conceptual challenges typically arise to keep students from thinking productively in this area? Various resources, such as Owens (1993), summarize researchers' answers to these questions. From the answers can come analogues to our

- Can you find a rule or relationship here?
- How does the rule work, and how is it helpful?
- Why does the rule work the way that it does?
- How are things changing?
- Is information given that lets me predict what is going to happen?
- Does my rule work for all cases?
- What steps am I doing over and over?
- Can I write down a mechanical rule that will do this job once and for all?
- How can I describe the steps without using specific inputs?
- When I do the same thing with different numbers, what still holds true? What changes?
- Now that I have an equation, how do the numbers (parameters) in the equation relate to the problem context?

Fig. 5 Sample questions for building rules to represent functions

set of algebraic-thinking guidelines and indicators of common pitfalls in students' thinking. These guidelines and indicators are the raw material for criteria to allow students' work to act as a lens on mathematical thinking.

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