

Numbers 01

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The Internet is a cornucopia of resources for learning and teaching math. We cruise bounteous bunches of Internet sites. In this document you'll find links to sites we like.

THIS UNIT IS ABOUT: numbers, natural numbers, counting numbers, whole numbers, integers, odd numbers, even numbers, prime numbers, and composite numbers.

Before we begin, here is a story about counting sheep.

The FIRST COMPUTER

by The Dragon

People's Computer Company Nov/Dec 1976

ONCE upon a time, thousands of years ago, an owner of sheep sat quietly, gazing upon her flock. Once the flock had been small and, every few days, she would match fingers to sheep, holding out one finger for each sheep. At the end of the day, when the flock owners came together by the firelight, frequently this one, then that one, then another, would hold up her hands, showing the number of fingers that corresponded to sheep.

Time passed, and the flocks prospered. On this day, the day of our story, she was troubled. She had matched fingers and sheep. All her fingers were extended, yet there were sheep for which there were no fingers. She tried again, taking the sheep in a different order, for they all had names. Still, with all fingers extended, there were sheep not included. She remembered that, recently, when flock owners gathered in the evening, others showed all fingers of both hands. She wondered if perhaps they also had sheep for which there were no matching fingers.

For a long time she sat quietly, thinking on the problem. Slowly, an idea began to form. She picked up a small stone and gazed at it intently for a long time. Then a smile burst unto her face and, without further hesitation, she again looked at her flock. On by one, she called out the names of her sheep and for each one added a stone to growing pile. Soon she was done, and for each sheep she held a stone in her hand.

That night, as the sky darkened and the fires were lighted, she could scarcely contain her excitement. The flock owners came together and, each in turn, told of her flocks. When her turn came, she took out a small bundle, the stones wrapped in a small sheepskin. Carefully, she spread the skin and arranged the stones. "Behold!" she cried, "This is my flock. For each sheep, a stone. For each stone, a sheep."

She explained her method. At first they were stunned ... then comprehension dawned and her smile grew into a circle of smiles around the fire.

And so, the flock owners adopted her method. Time passed, and the flocks prospered. For each sheep there was a stone, for each stone a sheep.

The piles of stones grew higher and higher. Then one day another idea But that's another story for another time.

Natural numbers, also called counting numbers

Natural numbers	1, 2, 3, 4, 5, 6, 7, 8, ...
Counting numbers	1, 2, 3, 4, 5, 6, 7, 8, ...
Odd natural numbers	1, 3, 5, 7, ...
Even natural numbers	2, 4, 6, 8, ...

Natural number – Wikipedia (http://en.wikipedia.org/wiki/Natural_number)

Whole numbers

Natural numbers	1, 2, 3, 4, 5, 6, 7, 8, ...
Whole numbers	0, 1, 2, 3, 4, 5, 6, 7, 8, ...
Odd whole numbers	1, 3, 5, 7, ...
Even whole numbers	0, 2, 4, 6, 8, ...

Whole number – Wikipedia (http://en.wikipedia.org/wiki/Whole_number)

Integers and assorted flavors of integers

Natural numbers	1, 2, 3, 4, ...
Whole numbers	0, 1, 2, 3, 4, ...
Integers	..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...
Positive integers	1, 2, 3, 4, ...
Nonnegative integers	0, 1, 2, 3, 4, ...
Negative integers	..., -4, -3, -2, -1
Nonpositive integers	..., -4, -3, -2, -1, 0
Odd integers	..., -3, -1, 1, 3, ...
Even integers	..., -4, -2, 0, 2, 4, ...

Integer – Wikipedia (<http://en.wikipedia.org/wiki/Integer>)

Prime numbers

- In the definitions below, you can replace **factor** by **divisor**.

Definition 1	A prime number is a natural number <i>greater than</i> 1 whose only natural number factors are 1 and the number itself.
Definition 2	A prime number is a natural number (positive integer) that has exactly two different natural number (positive integer) factors.
Bad definition	A prime number is a number that is evenly divisible only by itself and 1. [Oops. By this definition, 1 is prime. Not so.]
Why is 1 not prime?	www.utm.edu/research/primes/notes/faq/one.html
Online definitions	http://primes.utm.edu/glossary/page.php?sort=Prime http://en.wikipedia.org/wiki/Prime_number
The first 25 prime numbers	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
The first 1,000 primes The first 10,000 primes	http://primes.utm.edu/lists/small/1000.txt http://primes.utm.edu/lists/small/10000.txt
Amazing alakazams about prime numbers: <ul style="list-style-type: none"> • 1 is not a prime number. It is not greater than 1. It does not have two different factors. The one and only factor of 1 is 1. "One is one and all alone, and evermore shall be so." – Folk song <i>Green Grow the Rushes</i> • 2 is a prime number. It is greater than 1. It is divisible by 1 and itself. It has exactly two different factors, 1 and 2. It is the smallest prime number and the only even prime number. • 3 is a prime number. It is greater than 1. It is divisible by 1 and itself. It has exactly two different factors, 1 and 3. It is the smallest odd prime number. • 4 is not a prime number. It has three different factors, 1, 2, and 4. • 5 is a prime number. It is greater than 1. It is divisible by 1 and itself. It has exactly two different factors, 1 and 5. • 6 is not a prime number. It has four different factors, 1, 2, 3, and 6. • A proper factor of a natural number is a factor that is less than the number. • A prime number has exactly one natural number proper factor. The one and only natural number proper factor of a prime number is 1. • If a natural number is greater than 1 and is not a prime number, then it is a composite number. 	

Composite numbers	
<ul style="list-style-type: none"> In the definitions below, you can replace factor by divisor. 	
Definition 1	A composite number is a natural number that has three or more different natural number factors.
Definition 2	A composite number is a natural number greater than 1 that has natural number factors other than 1 and the number itself.
Definition 3	A composite number is a natural number that is neither 1 nor a prime number.
The first 73 composite numbers	4, 6, 8, 9, 10, 12, 14, 15, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 49, 50, 51, 52, 54, 55, 56, 57, 58, 60, 62, 63, 64, 65, 66, 68, 69, 70, 71, 72, 74, 75, 76, 77, 78, 80, 81, 82, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 98, 99
Go fish	http://en.wikipedia.org/wiki/Composite_number http://primes.utm.edu/glossary/page.php?sort=Composite
Amazing alakazams about composite numbers: <ul style="list-style-type: none"> 1 is not a composite number. It has a paucity of factors. The one and only factor of 1 is 1. 2 is not a composite number. It has exactly two different factors, 1 and 2. 3 is not a composite number. It has exactly two different factors, 1 and 3. 4 is a composite number. It has three different factors, 1, 2, and 4. 5 is not a composite number. It has exactly two different factors, 1 and 5. 6 is composite number. It has four different factors, 1, 2, 3, and 6. 8 is a composite number. It has four different factors, 1, 2, 4, and 8. A proper factor of a natural number is a factor that is less than the number. A prime number has one proper factor. The only proper factor of a prime number is 1. A composite number has 2 or more different natural number proper factors. The proper factors of 4 are 1 and 2. The proper factors of 6 are 1, 2, and 3. The proper factors of 8 are 1, 2 and 4. If a natural number is greater than 1 and is not a composite number, then it is a prime number. 	
Natural numbers	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, ...
Prime numbers	2, 3, 5, 7, 11, 13, ...
Composite numbers	4, 6, 8, 10, 12, ...

Testing for prime numbers in natural number land

There are three types of natural numbers: the number 1, prime numbers, and composite numbers.

Let's start small. Assume that N is a natural number greater than 1 and less than 100. Is N a prime number? If N is not 1 and not a prime number, then N is a composite number.

N is one of these numbers: 1, 2, 3, 4, 5, ..., 98, 99.

- If $N = 1$, then N is not a prime number and N is not a composite number.
- If $N = 2$, then N is a prime number.
- If $N > 2$, use one of the methods below to determine if it is a prime number or a composite number.

The Brute Force Method: Divide N by 2, 3, 4, 5, 6, ..., $N - 1$. If none of these divide evenly into N , then N is prime. To determine that 97 is prime requires 95 divisions. Totally tiresome!

A Better Brute Force Method. If $N > 2$ and N is even, then N is not a prime number. If N is odd, divide N by 3, 5, 7, 9, 11, ..., $N - 2$. If none of these divide evenly into N , then N is prime. To determine that 97 is prime requires 47 divisions. About half as much work as the brute force method described previously.

A Much Better Brute Force Method. If $N = 3$, it is prime. Suppose $N > 3$. If N is even, then N is not prime. If N is odd, divide by **odd integers** from 3 up to the integer part of the square root of N . That is, divide by 3, 5, 7, 9, 11, ..., $\text{int}(\sqrt{N})$. Example: $\sqrt{97} = 9.8488\dots$, so the integer part of $\sqrt{97}$ is 9. To determine that 97 is prime, you need divide by only 3, 5, 7, and 9.

Even Better Brute Force Method. Assume $N > 3$. If N is even, then N is not prime. If N is odd, divide by **odd prime numbers** less than or equal to the integer part of the square root of N . For example, the integer part of $\sqrt{97}$ is 9. To determine that 97 is prime, you need divide by only the primes 3, 5, and 7.

If you memorize the 25 primes that are less than 100, you can determine whether any number less than 10,000 is a prime number with at most 25 divisions. Why?

Use divisibility rules

- If N is an even number greater than 2, it is divisible by 2 and is not prime.
- If $N > 3$ and the sum of the digits of N is divisible by 3, then N is not prime.
- If $N > 5$ and ends in 0 or 5, then N is not prime.

Et cetera, et cetera, et cetera. Go fish:

Divisibility rule – Wikipedia (http://en.wikipedia.org/wiki/Divisibility_rule)

BEYOND HERE THIS DOCUMENT IS UNDER CONSTRUCTION

Check out the Sieve of Eratosthenes.

- Sieve of Eratosthenes NIST (www.nist.gov/dads/HTML/sieve.html)
- Eratosthenes (www.math.utah.edu/~alfeld/Eratosthenes.html)

Conjecture: Every prime > 5 is of the form $6k + 1$ or $6k + 5$, where k is a natural number.

- Is this true? If yes, can you prove it?

If n is a prime number greater than 5, then there exists a natural number k such that $n = 6k + 1$ or $n = 6k + 5$. In other words, every prime number n greater than 5 is of one of the following forms:

- $n = 6k + 1$ or $n = 6k + 5$

Let's try it for a few primes.

- If $n = 7$, then $k = 1$: $7 = 6(1) + 1$. Yup.
- If $n = 11$, then $k = 1$: $11 = 6(1) + 5$. Right on.
- If $n = 13$, then $k = 2$: $13 = 6(2) + 1$. Lookin' good.
- If $n = 17$, then $k = 2$: $17 = 6(2) + 5$. AOK.
- Et cetera, et cetera, et cetera???

So, tra la, tra la, pretend that the conjecture is true. If you divide a natural number greater than 5 by 6 and the remainder is 1 or 5, then the number **might be prime**. If the remainder is 0, 2, 3, or 4, the number is definitely **not prime**.

Is 7081 prime? Divide 7081 by 6. We get quotient = 1180 and remainder = 1, so 7081 **might be prime**. 7081 is not divisible by 2, 3, or 5. Continue dividing 7081 by primes less than or equal to the integer part of the square root of 7081. $\text{int}(\sqrt{7081}) = 84$.

Oh, you don't know primes greater than 5? That's OK -- divide 7081 by numbers that might be primes. These are numbers of the form $6k + 1$ and $6k + 5$ for $k = 1, 2, 3$, et cetera. Feel free to omit any divisor that you recognize as not prime.

k	$6k + 1$	$6k + 5$		k	$6k + 1$	$6k + 5$
1	7	11		6	37	41
2	13	17		7	43	47
3	19	23		8	49	53
4	25	29		9	55	59
5	31	35		etc		

Now try this one: Is 6497 prime?

Crunch numbers cleverly and prime time dragons will dance on your keyboard.

GLOSSARY

counting number, *n*. A counting number as a **natural number** or **positive integer**. Counting numbers are the numbers 1, 2, 3, and so on. However, we have seen definitions that included zero as a counting number. For more about this, we recommend a visit to

divisor, *n*. A divisor of a number is a natural number that divides the given number evenly with a remainder of zero. The number 1 is a divisor of every number. The number 2 is a divisor of every even number. A **factor** of a natural number is a divisor of the number

Division (mathematics) – Wikipedia

(http://en.wikipedia.org/wiki/Division_%28mathematics%29)

factor, *n*. If you multiply two or more **natural numbers**, the **product** is a natural number. The numbers you multiplied to obtain the product are factors of the product. For example:

- $1 \times 2 = 2$, so 1 and 2 are factors of 2
- $3 \times 5 = 15$, so 3 and 5 are factors of 15.
- $2 \times 3 \times 4 = 24$, so 2, 3, and 4 are factors of 24

factor pair, *n*. If you multiply two **natural numbers**, the **product** is a natural number. The two numbers you multiplied to obtain the product are a factor pair of the product. For example:

Factor pairs of 6:

- $1 \times 6 = 6$. 1 and 6 are a factor pair of 6.
- $2 \times 3 = 6$. 2 and 3 are a factor pair of 6.
- The factors of 6 are 1, 2, 3, and 6.

Factor pair of 7:

- $1 \times 7 = 7$. 1 and 7 are the only factor pair of 7.

Factor pairs of 12:

- $1 \times 12 = 12$. 1 and 12 are a factor pair of 12.
- $2 \times 6 = 12$. 2 and 6 are a factor pair of 12.
- $3 \times 4 = 12$. 3 and 4 are a factor pair of 12.
- The factors of 12 are 1, 2, 3, 4, 6, and 12.

Here are some amazing alakazams about **natural numbers** and factors:

1 has only one factor. The only factor of 1 is itself, 1. **1** is the only natural number that has only one factor and 1 is a factor of every natural number. Here are some words from an old song about numbers.

I'll sing you one-o, green grow the rushes-o.
What is your one-o?
One is one and all alone and evermore shall be so.

2 has two factors. The factors of 2 are 1 and 2. A natural number that has exactly two different factors is a **prime number**. **2** is the smallest prime number and the only even prime number.

3 has two factors. The factors of 3 are 1 and 3. **3** is a **prime number** because it has exactly two different factors. It is the smallest odd prime number.

4 has three factors. The factors of 4 are 1, 2, and 4. If a natural number is not 1 and is not a prime number, it is a **composite number**. **4** is the smallest composite number and the smallest natural number that has exactly three factors.

5 has two factors. The factors of 5 are 1 and 5. Yes: **5** is a prime number.

6 has four factors. The factors of 6 are 1, 2, 3, and 6. **6** is a composite number and the smallest natural number that has exactly four factors.

7 has two factors. The factors of 7 are 1 and 7. What kind of number is 7? Answer: **prime number**.

8 has four factors. The factors of 8 are 1, 2, 4, and 8. What kind of number is 8? Answer: **composite number**.

Above we mentioned three kinds of natural numbers.

- 1 (One is one and all alone and evermore shall be so.)
- prime numbers
- composite numbers

Your Turn.

1. What are the factors of 9? What kind of number is 9? Answer: 1, 3, and 9; **composite number**.
2. What are the factors of 10? What kind of number is 10? Answer: 1, 2, 5, and 10; **composite number**.
3. What are the factors of 11? What kind of number is 11? Answer: 1 and 11; **prime number**.
4. What natural number is a factor of every natural number? Answer: 1.
5. What is the smallest factor of any natural number? Answer: 1.
6. What is the largest factor of any natural number? Answer: the number itself.
7. What is the smallest natural number that has exactly five factors? What are the factors? Answer: 16; 1, 2, 4, 8, and 16.
8. What is the smallest natural number that has exactly six factors? What are the factors? Answer: 12; 1, 2, 3, 4, 6, and 12.
9. What is the smallest natural number that has exactly 11 factors? What are the factors? Answer: 1024; 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, and 1024.
10. What is the smallest natural number that has exactly 16 factors? What are the factors? Answer: 120; 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120.
11. What natural number has exactly one factor? Answer: 1.
12. What kind of natural number has exactly two different factors? Answer: **prime number**.

13. What kind of natural number has three or more different factors? Answer: **composite number**.

integer, n . An integer is a **natural number**, zero, or the negative (opposite) of a natural number. You may see the integers described in a sentence like this:

- The integers are the numbers $0, \pm 1, \pm 2, \pm 3, \dots$

The three dots following " ± 3 ," are an *ellipsis* and mean "and so on." The fourth dot is the period that ends the sentence. So you may read the sentence as, "The integers are the numbers $0, \pm 1, \pm 2, \pm 3$, and so on."

You may see the integers described as shown below with a left-hand ellipsis indicating that the numbers go on in the negative direction and a right-hand ellipsis indicating that the numbers go on in the positive direction:

- The integers are the numbers $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$

One way to read the sentence is, "The integers are the numbers $-1, -2, -3$, and so on, 0 , and $1, 2, 3$, and so on."

Here's another definition of an integer: An integer is any number that can be obtained by adding or subtracting two **natural numbers**.

- How do you get 0 ? Pick any natural number and subtract it from itself. For example, $1 - 1 = 0$.
- How about -1 ? Easy: subtract 2 from 1 , like this: $1 - 2 = -1$.
- To get 2 , add 1 to 1 : $1 + 1 = 2$.
- Now that we have 2 , add 1 to get 3 : $2 + 1 = 3$.
- And so on. (\dots)

Integers come in three flavors:

- Positive integers ($1, 2, 3, \dots$), also called **natural numbers** or **counting numbers**. All positive integers are greater than zero.
- Zero (0).
- Negative integers ($\dots, -3, -2, -1$). These are the opposites (negatives) of the positive integers. All negative integers are less than zero.

natural number, n . The natural numbers are $1, 2, 3$, and so on forever. They keep going and going and going. If you pick any natural number, we can pick a bigger one. Easy – we pick the natural number that is 1 more than the natural number you picked. If you pick 1 , we pick 2 . If you pick 2 , we pick 3 . If you grab 100 , we latch on to 101 . If you bodaciously choose a *secret* natural number, we admire your cleverness and say, "Our natural number is your secret natural number plus 1 ." Natural numbers are also called **counting numbers** and **positive integers**.

In a math book, you might see the natural numbers described in a sentence like this:

- The natural numbers are 1, 2, 3, 4, 5,

The three dots following "3," are called an *ellipsis* and mean "and so on." The fourth dot is the period that ends the sentence. So you may read the sentence as, "The natural numbers are 1, 2, 3, and so on."

positive integer, n . The positive integers are the numbers 1, 2, 3, and so on. They are also called **natural numbers** or **counting numbers**.

prime number, n . A prime number is a **natural number (positive integer)** that has exactly two distinct **factors**. The factors of a prime number are 1 and the number itself.

- **1** is not a prime number because it has only one factor. The one and only factor of 1 is 1.
- **2** is a prime number. It has exactly two distinct factors, 1 and 2. It is the smallest prime number and the only even prime number.
- **3** is a prime number. It has exactly two distinct factors, 1 and 3. It is the smallest odd prime number.
- **4** is not a prime number because it has three distinct factors, 1, 2, and 4. **4** is a **composite number**.
- **5** is a prime number. It has exactly two distinct factors, 1 and 5.
- **6** is not a prime number because it has four distinct factors, 1, 2, 3, and 6. **6** is a **composite number**.
- The first 25 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.
- A prime number has exactly one **proper factor**. The one and only proper factor of a prime number is 1.
- If a **natural number** is greater than 1 and is not a prime number, then it is a **composite number**.

proper factor, n . A proper factor of a **natural number** is any **factor** other than the number itself.

- **1** doesn't have any proper factors. The only factor of 1 is itself, 1.
- **2** has one proper factor. The one and only proper factor of 2 is 1. A **natural number** that has exactly one proper factor is a **prime number**. Please note that 1 is not a prime number because it has no proper factor.
- **3** has one proper factor. The one and only proper factor of 3 is 1. Yes, 3 is a **prime number**.
- **4** has two proper factors. The proper factors of 4 are 1 and 2. A **natural number** that has two or more proper factors is a **composite number**.
- **5** has one proper factor. The one and only proper factor of 5 is 1. What kind of number is 5? Answer: **prime number**.
- **6** has three proper factors. The proper factors of 6 are 1, 2, and 3. What kind of number is 6? Answer: **composite number**.
- **7** has one proper factor. The one and only proper factor of 7 is 1. What kind of number is 7? Answer: **prime number**.

- **8** has three proper factors. What are the proper factors of 8? What kind of number is 8? Answer: 1, 2, and 4; **composite number**.

Above we mentioned three kinds of **natural numbers**.

- 1 has no proper factor.
- A **prime number** has exactly one proper factor.
- A **composite number** has two or more different proper factors.

Your Turn.

1. What are the proper factors of 9? What kind of number is 9? Answer: 1 and 3; **composite number**.
2. What are the proper factors of 10? What kind of number is 10? Answer: 1, 2, and 5; **composite number**.
3. What is the proper factor of 11? What kind of number is 11? Answer: 1; **prime number**.
4. What are the proper factors of 12? What kind of number is 12? Answer: 1, 2, 3, 4, and 6; **composite number**.
5. What is the only proper factor of a prime number? Answer: 1.
6. What is the smallest proper factor of a composite number? Answer: 1.
7. What is the smallest natural number that has exactly two proper factors? What are the proper factors? Answer: 4; 1 and 2.
8. What is the smallest natural number that has exactly three proper factors? What are the proper factors? Answer: 6; 1, 2, and 3.
9. What is the smallest natural number that has exactly 10 proper factors? What are the proper factors? Answer: 1024; 1, 2, 4, 8, 16, 32, 64, 128, 256, and 512.
10. What is the smallest natural number that has exactly 14 proper factors? What are the proper factors? Answer: 120; 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, and 60.
11. What natural number has no proper factor? Answer: 1.
12. What kind of natural number has exactly one proper factor? Answer: prime number.
13. What kind of natural number has two or more proper factors? Answer: composite number.

whole number, n. In various math textbooks and math dictionaries that lurk about our premises, we can find different definitions of a whole number. Here are two incompatible definitions:

- **Definition 1:** The whole numbers are 0 and the **natural numbers (positive integers)**. According to this definition, the whole numbers are 0, 1, 2, 3, and so on. **We'll go with this definition.**
- **Definition 2:** A whole number is a number having a fractional part equal to zero. According to this definition, the set of whole numbers is the set of **integers**: ..., -3, -2, -1, 0, 1, 2, 3, We don't like this definition.

zillion, n. The number of seconds it will take us to finish konstruktion of this stuff and fixx all of the mistrakes.