

## 13.12 Solved Problems

### 13.12.1 Plane Electromagnetic Wave

Suppose the electric field of a plane electromagnetic wave is given by

$$\vec{\mathbf{E}}(z,t) = E_0 \cos(kz - \omega t) \hat{\mathbf{i}} \quad (13.12.1)$$

Find the following quantities:

- The direction of wave propagation.
- The corresponding magnetic field  $\vec{\mathbf{B}}$ .

#### Solutions:

(a) By writing the argument of the cosine function as  $kz - \omega t = k(z - ct)$  where  $\omega = ck$ , we see that the wave is traveling in the  $+z$  direction.

(b) The direction of propagation of the electromagnetic waves coincides with the direction of the Poynting vector which is given by  $\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{B}} / \mu_0$ . In addition,  $\vec{\mathbf{E}}$  and  $\vec{\mathbf{B}}$  are perpendicular to each other. Therefore, if  $\vec{\mathbf{E}} = E(z,t) \hat{\mathbf{i}}$  and  $\vec{\mathbf{S}} = S \hat{\mathbf{k}}$ , then  $\vec{\mathbf{B}} = B(z,t) \hat{\mathbf{j}}$ . That is,  $\vec{\mathbf{B}}$  points in the  $+y$ -direction. Since  $\vec{\mathbf{E}}$  and  $\vec{\mathbf{B}}$  are in phase with each other, one may write

$$\vec{\mathbf{B}}(z,t) = B_0 \cos(kz - \omega t) \hat{\mathbf{j}} \quad (13.12.2)$$

To find the magnitude of  $\vec{\mathbf{B}}$ , we make use of Faraday's law:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \quad (13.12.3)$$

which implies

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad (13.12.4)$$

From the above equations, we obtain

$$-E_0 k \sin(kz - \omega t) = -B_0 \omega \sin(kz - \omega t) \quad (13.12.5)$$

or

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c \quad (13.12.6)$$

Thus, the magnetic field is given by

$$\vec{\mathbf{B}}(z, t) = (E_0 / c) \cos(kz - \omega t) \hat{\mathbf{j}} \quad (13.12.7)$$

### 13.12.2 One-Dimensional Wave Equation

Verify that, for  $\omega = kc$ ,

$$\begin{aligned} E(x, t) &= E_0 \cos(kx - \omega t) \\ B(x, t) &= B_0 \cos(kx - \omega t) \end{aligned} \quad (13.12.8)$$

satisfy the one-dimensional wave equation:

$$\left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \begin{Bmatrix} E(x, t) \\ B(x, t) \end{Bmatrix} = 0 \quad (13.12.9)$$

#### Solution:

Differentiating  $E = E_0 \cos(kx - \omega t)$  with respect to  $x$  gives

$$\frac{\partial E}{\partial x} = -kE_0 \sin(kx - \omega t), \quad \frac{\partial^2 E}{\partial x^2} = -k^2 E_0 \cos(kx - \omega t) \quad (13.12.10)$$

Similarly, differentiating  $E$  with respect to  $t$  yields

$$\frac{\partial E}{\partial t} = \omega E_0 \sin(kx - \omega t), \quad \frac{\partial^2 E}{\partial t^2} = -\omega^2 E_0 \cos(kx - \omega t) \quad (13.12.11)$$

Thus,

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \left( -k^2 + \frac{\omega^2}{c^2} \right) E_0 \cos(kx - \omega t) = 0 \quad (13.12.12)$$

where we have made use of the relation  $\omega = kc$ . One may follow a similar procedure to verify the magnetic field.