

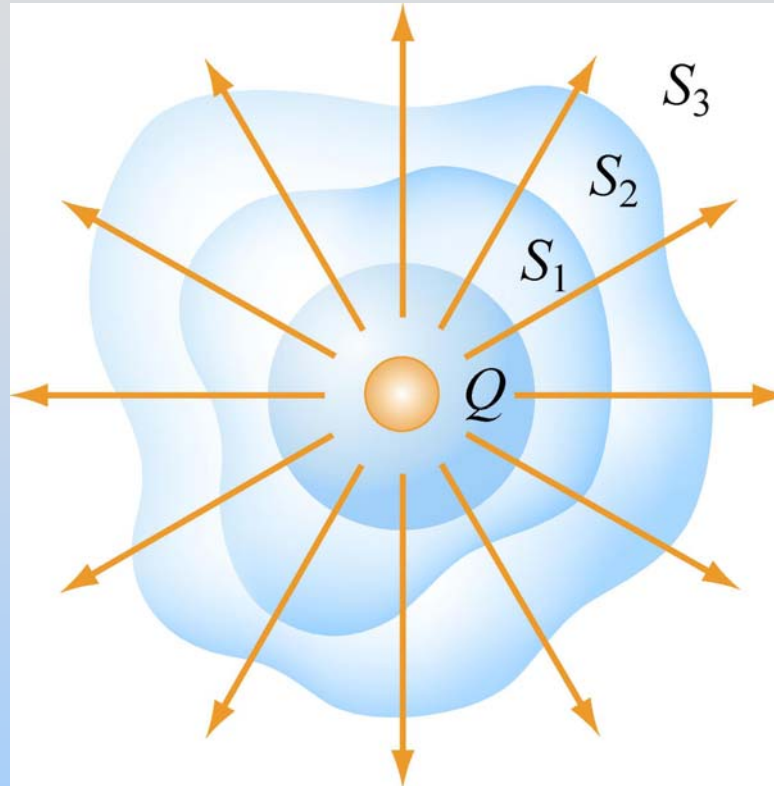
Gauss's Law

The first Maxwell Equation

A very useful computational technique

This is important!

Gauss's Law – The Idea



The total “flux” of field lines penetrating any of these surfaces is the same and depends only on the amount of charge inside

Gauss's Law – The Equation

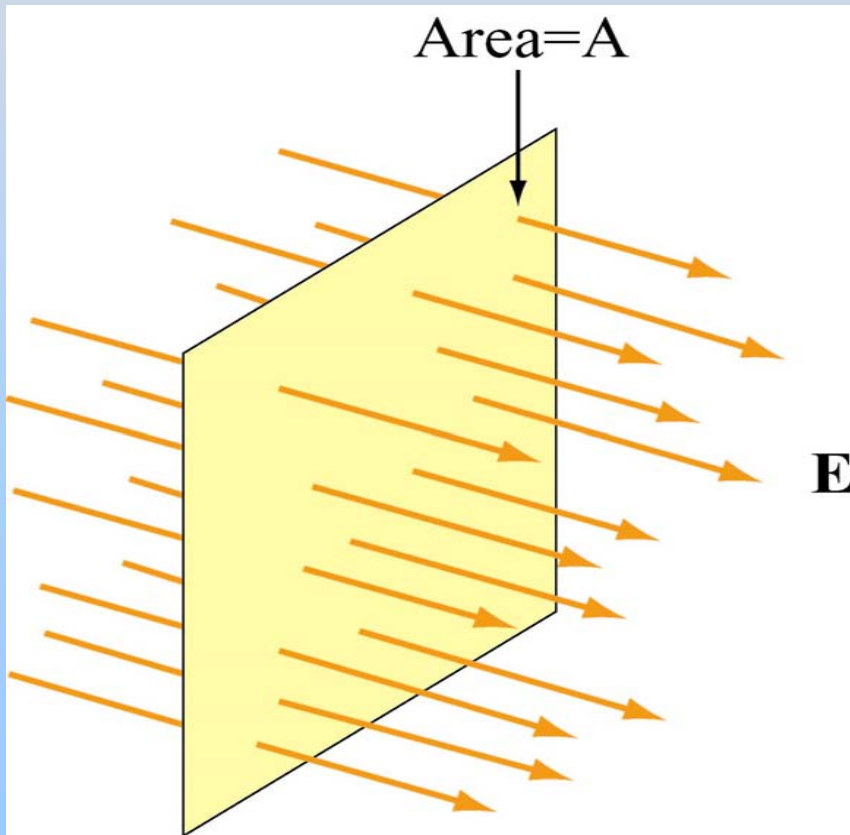
$$\Phi_E = \oiint_{\text{closed surface } S} \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}$$

Electric flux Φ_E (the surface integral of E over closed surface S) is proportional to charge inside the volume enclosed by S

Now the Details

Electric Flux Φ_E

Case I: E is constant vector field perpendicular to planar surface S of area A



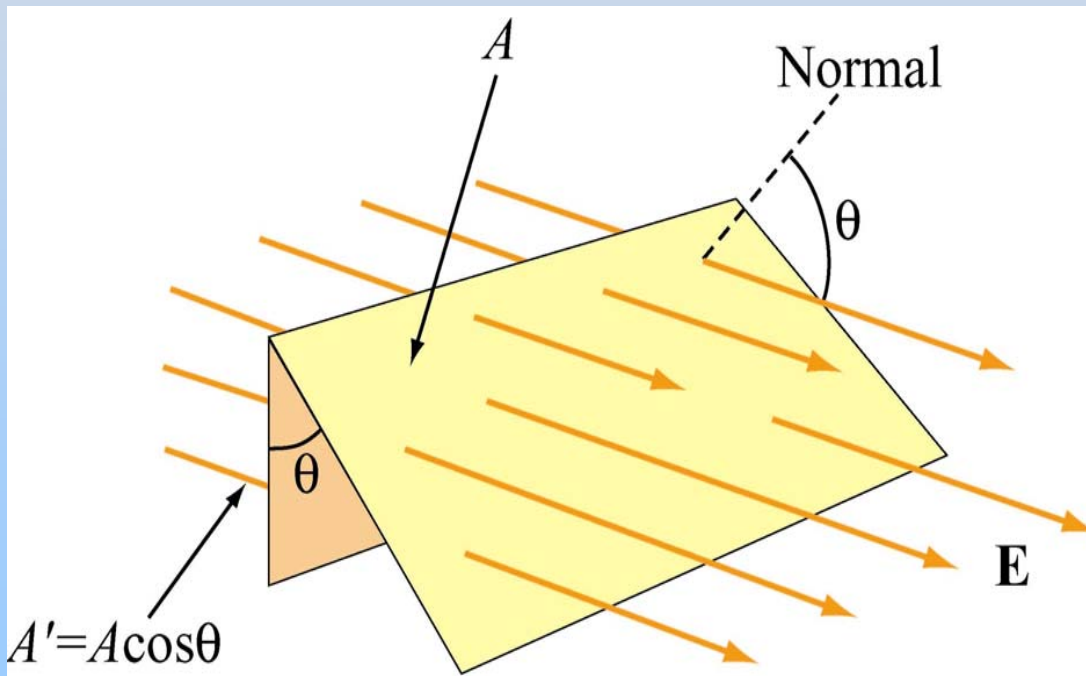
$$\Phi_E = \iint \vec{E} \cdot d\vec{A}$$

$$\Phi_E = +EA$$

Our Goal: Always reduce problem to this

Electric Flux Φ_E

Case II: \mathbf{E} is constant vector field directed at angle θ to planar surface S of area A



$$\Phi_E = \iint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$\Phi_E = EA \cos \theta$$

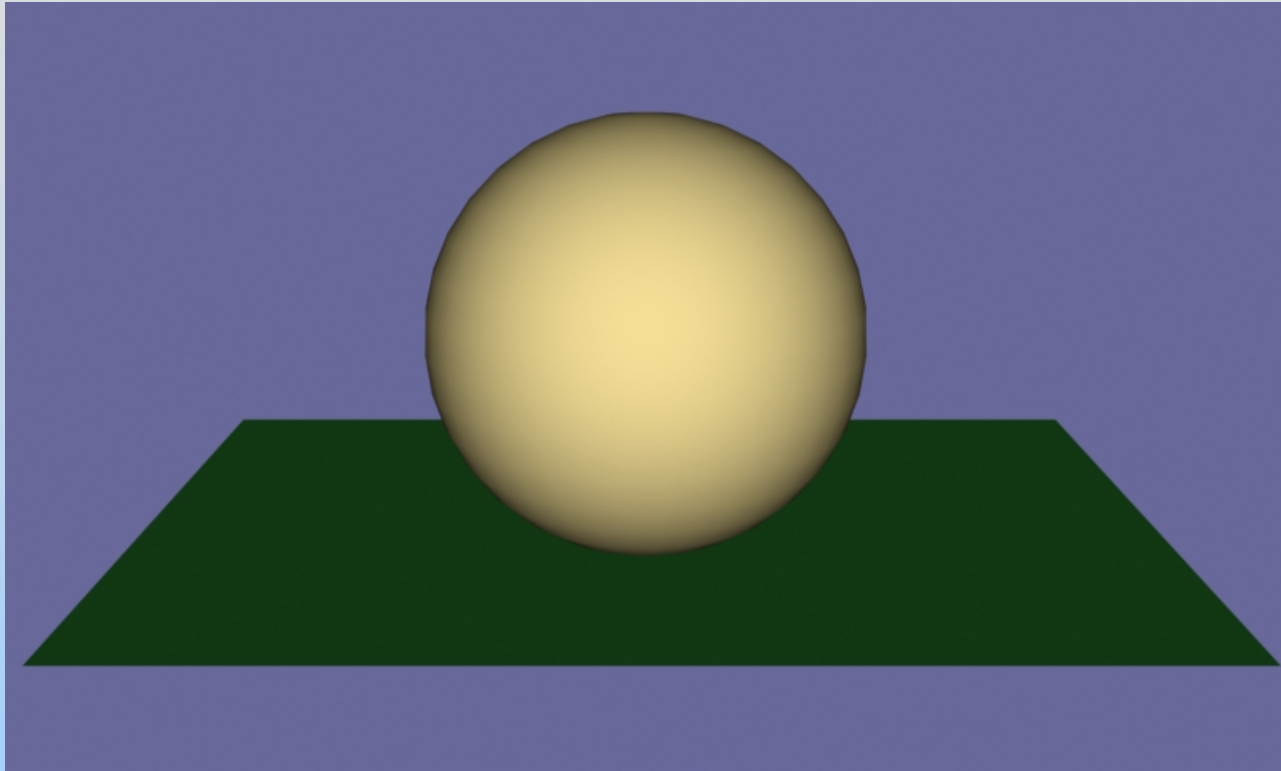
PRS Question: Flux Thru Sheet

Gauss's Law

$$\Phi_E = \oiint_{\text{closed surface } S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\epsilon_0}$$

Note: Integral must be over closed surface

Open and Closed Surfaces

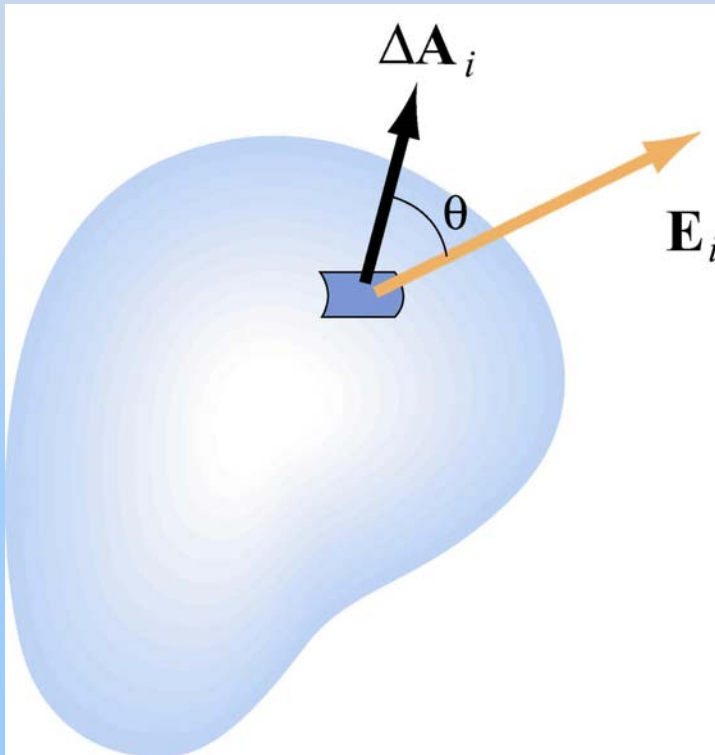


A rectangle is an open surface — it does NOT contain a volume

A sphere is a closed surface — it DOES contain a volume

Area Element $d\mathbf{A}$: Closed Surface

For closed surface, $d\mathbf{A}$ is normal to surface
and points outward
(from inside to outside)

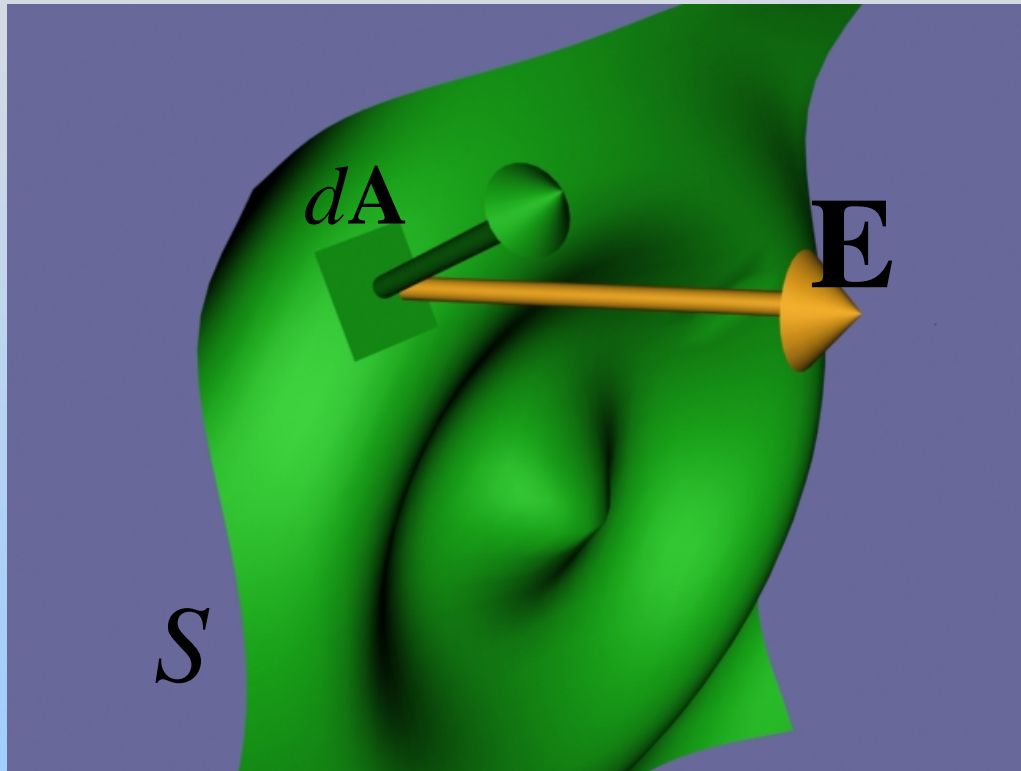


$\Phi_E > 0$ if \mathbf{E} points out

$\Phi_E < 0$ if \mathbf{E} points in

Electric Flux Φ_E

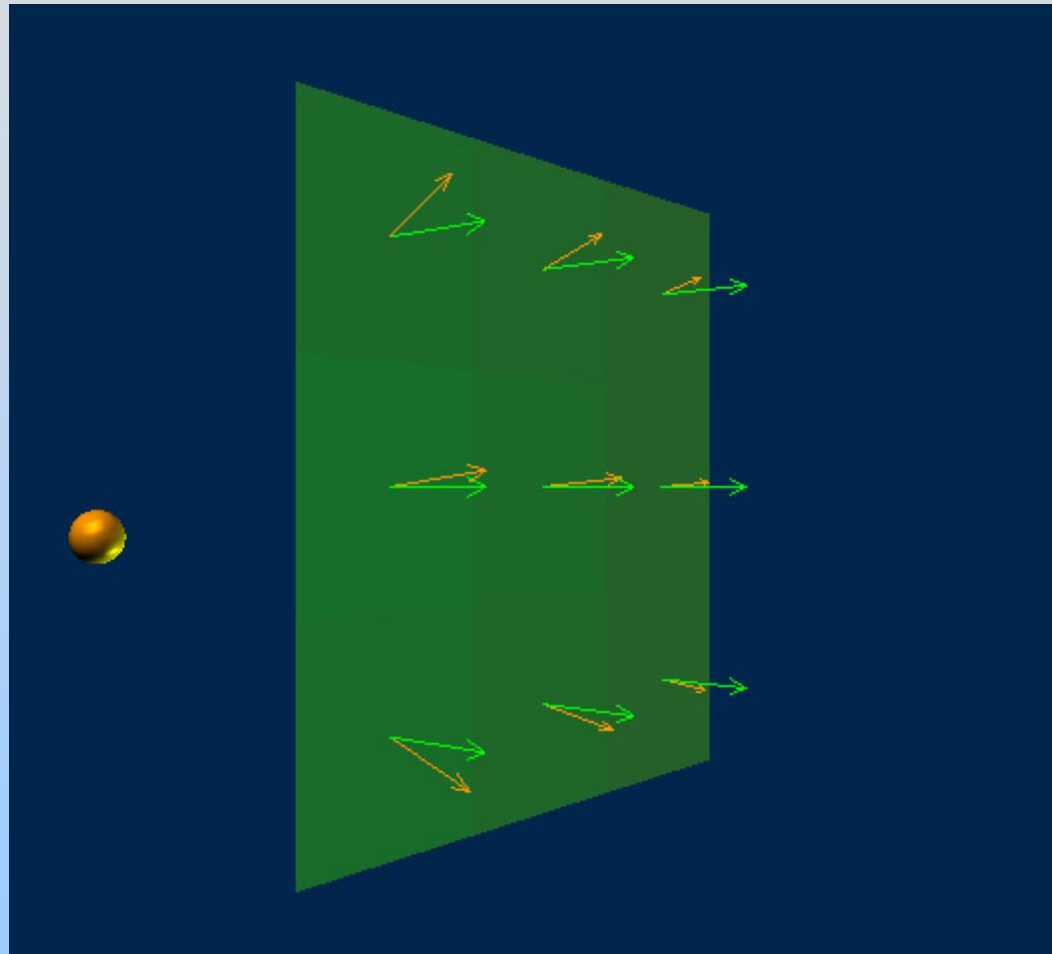
Case III: E not constant, surface curved



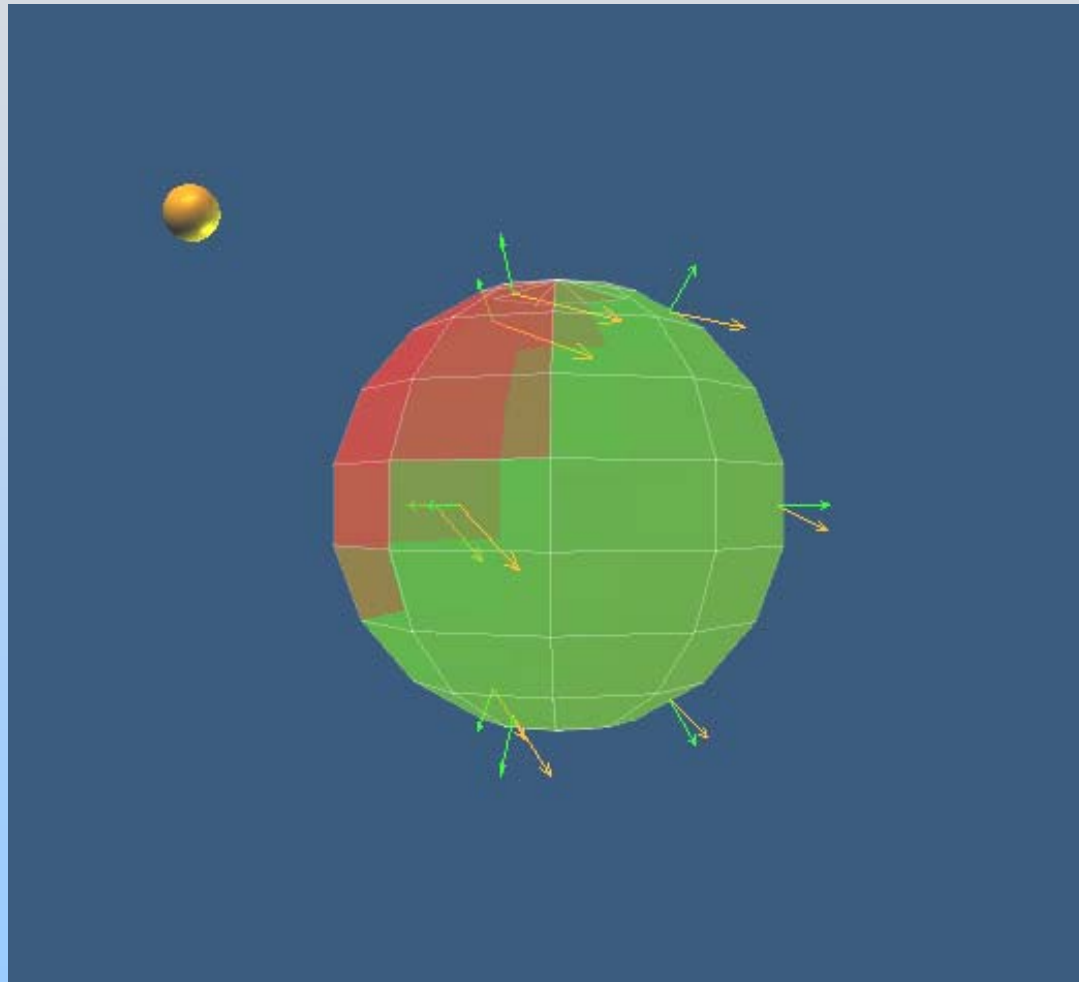
$$d\Phi_E = \vec{E} \cdot d\vec{A}$$

$$\Phi_E = \iint d\Phi_E$$

Example: Point Charge Open Surface



Example: Point Charge Closed Surface



PRS Question: Flux Thru Sphere

Electric Flux: Sphere

Point charge Q at center of sphere, radius r

E field at surface:

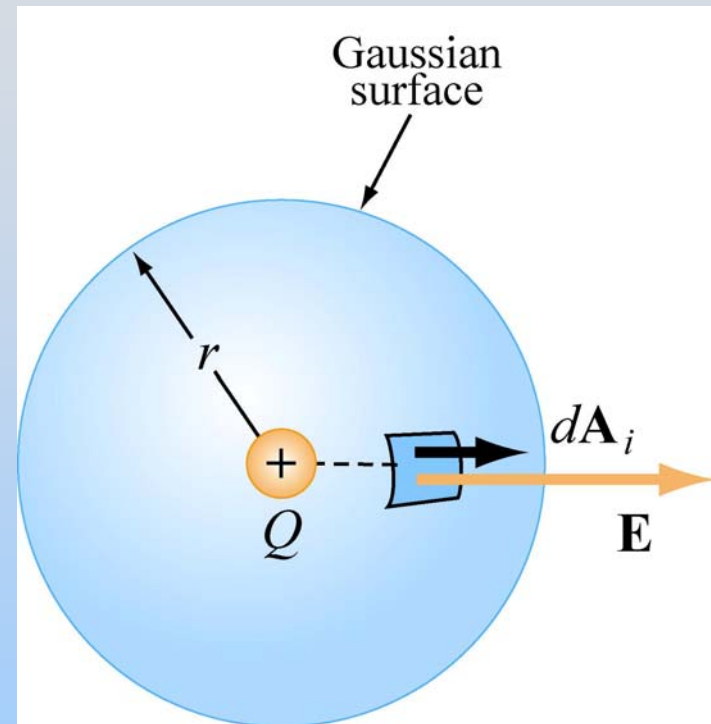
$$\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

Electric flux through sphere:

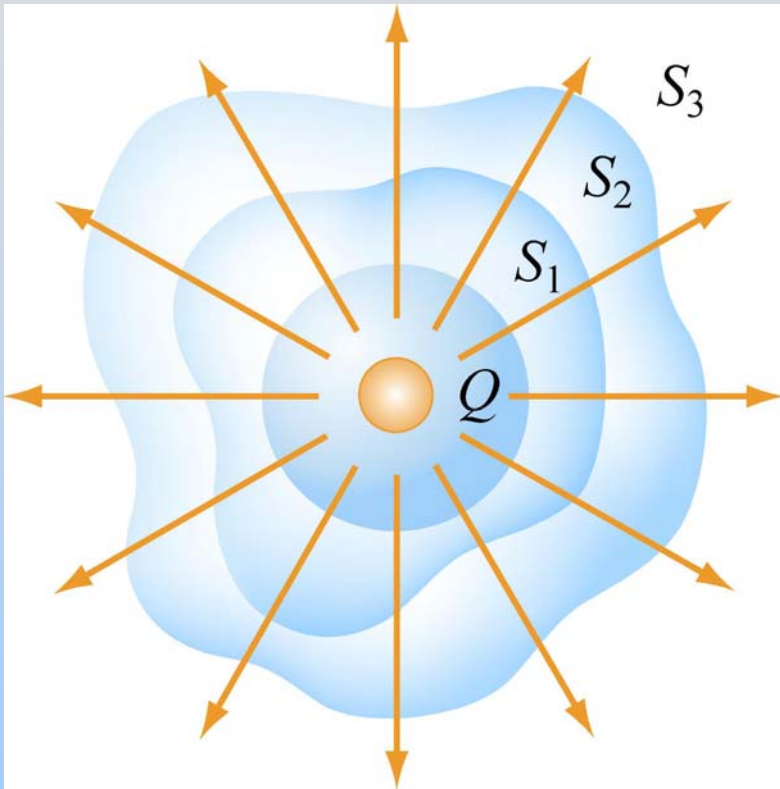
$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oiint_S \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \cdot dA \hat{\mathbf{r}}$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \oiint_S dA = \frac{Q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\boxed{d\vec{\mathbf{A}} = dA \hat{\mathbf{r}}}$$



Arbitrary Gaussian Surfaces



$$\Phi_E = \oiint_{\text{closed surface } S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\epsilon_0}$$

For all surfaces such as S_1 , S_2 or S_3

Applying Gauss's Law

1. Identify regions in which to calculate E field.
2. Choose Gaussian surfaces S: Symmetry
3. Calculate $\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$
4. Calculate q_{in} , charge enclosed by surface S
5. Apply Gauss's Law to calculate E:

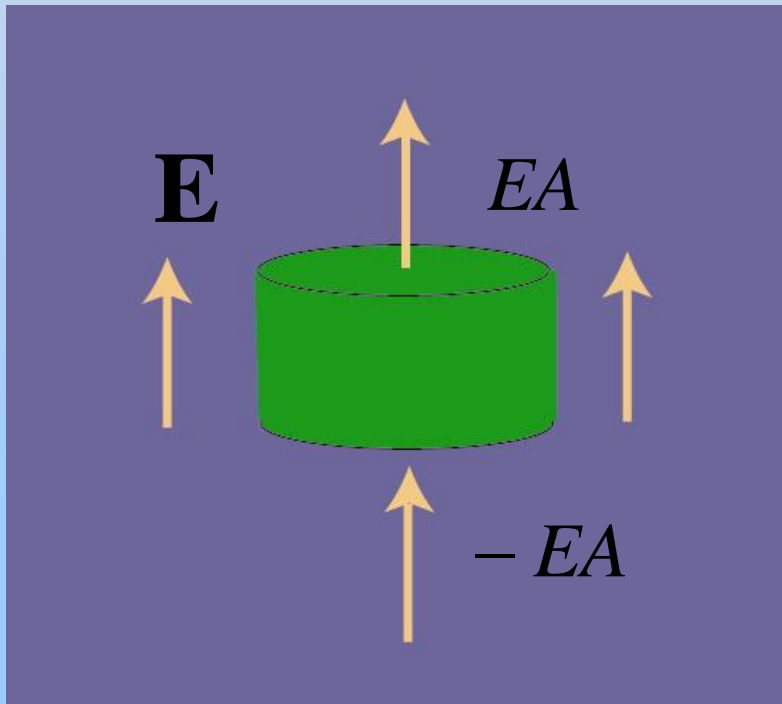
$$\Phi_E = \oiint_{\text{closed surface } S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{in}}{\epsilon_0}$$

Choosing Gaussian Surface

Choose surfaces where \mathbf{E} is perpendicular & constant.
Then flux is EA or $-EA$.

OR

Choose surfaces where \mathbf{E} is parallel.
Then flux is zero



Example: Uniform Field

Flux is EA on top
Flux is $-EA$ on bottom
Flux is zero on sides

Symmetry & Gaussian Surfaces

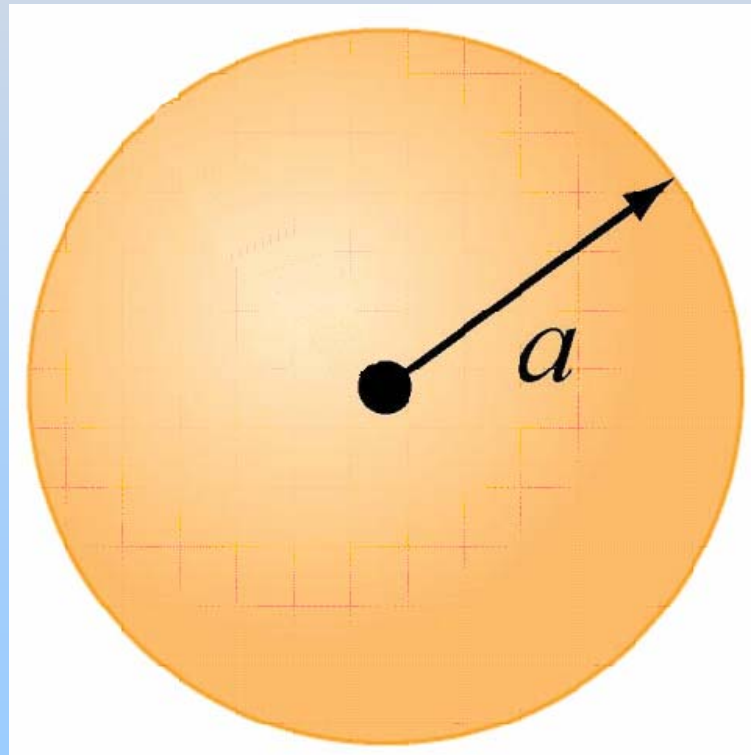
Use Gauss's Law to calculate E field from highly symmetric sources

Symmetry	Gaussian Surface
Spherical	Concentric Sphere
Cylindrical	Coaxial Cylinder
Planar	Gaussian "Pillbox"

**PRS Question:
Should we use Gauss' Law?**

Gauss: Spherical Symmetry

+ Q uniformly distributed throughout non-conducting solid sphere of radius a . Find \mathbf{E} everywhere

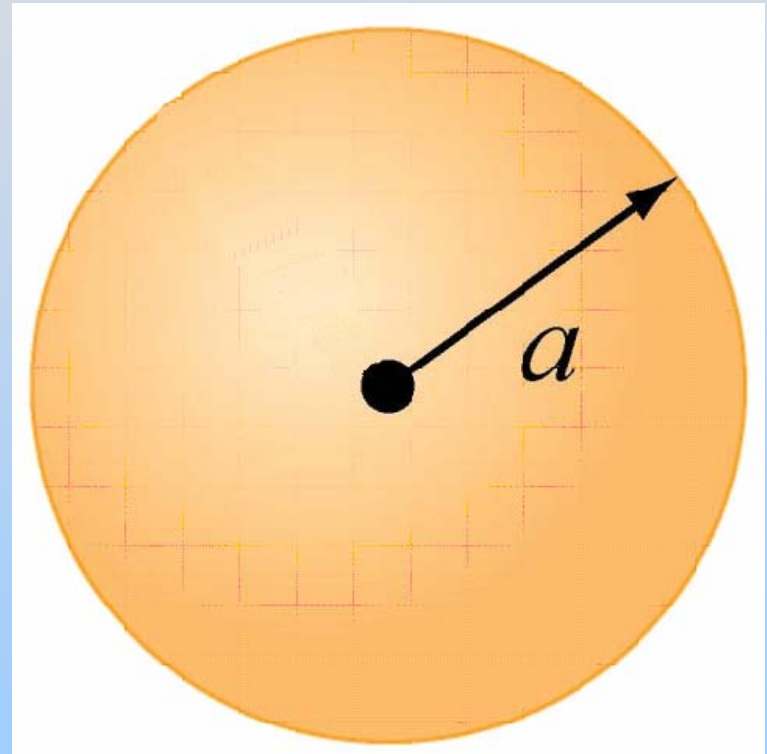


Gauss: Spherical Symmetry

Symmetry is Spherical

$$\vec{\mathbf{E}} = E \hat{\mathbf{r}}$$

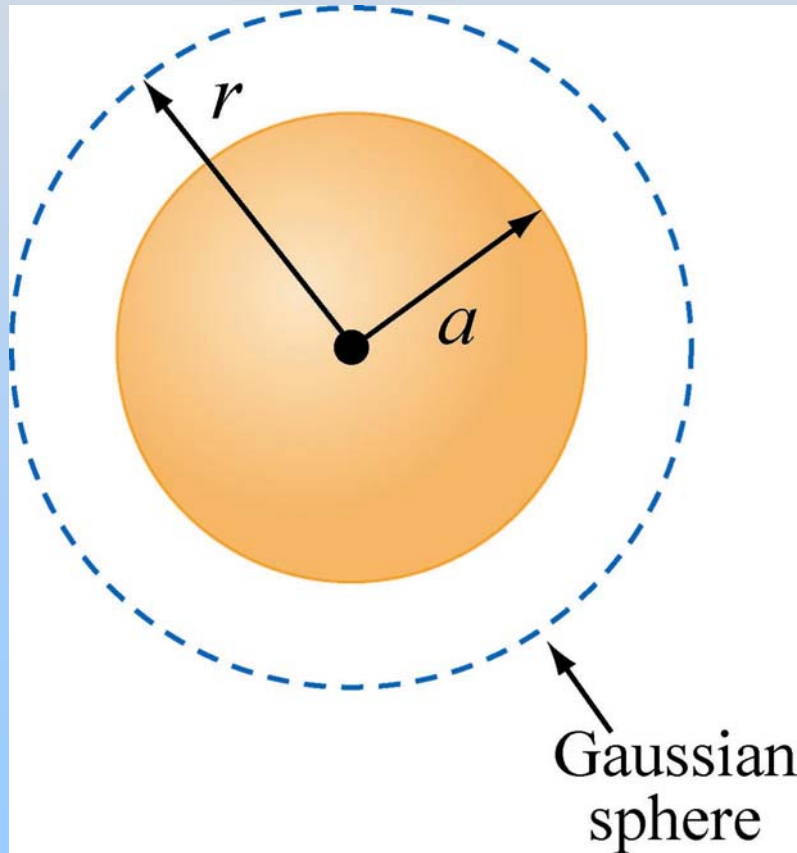
Use Gaussian Spheres



Gauss: Spherical Symmetry

Region 1: $r > a$

Draw Gaussian Sphere in Region 1 ($r > a$)



Note: r is arbitrary **but** is the radius for which you will calculate the E field!

Gauss: Spherical Symmetry

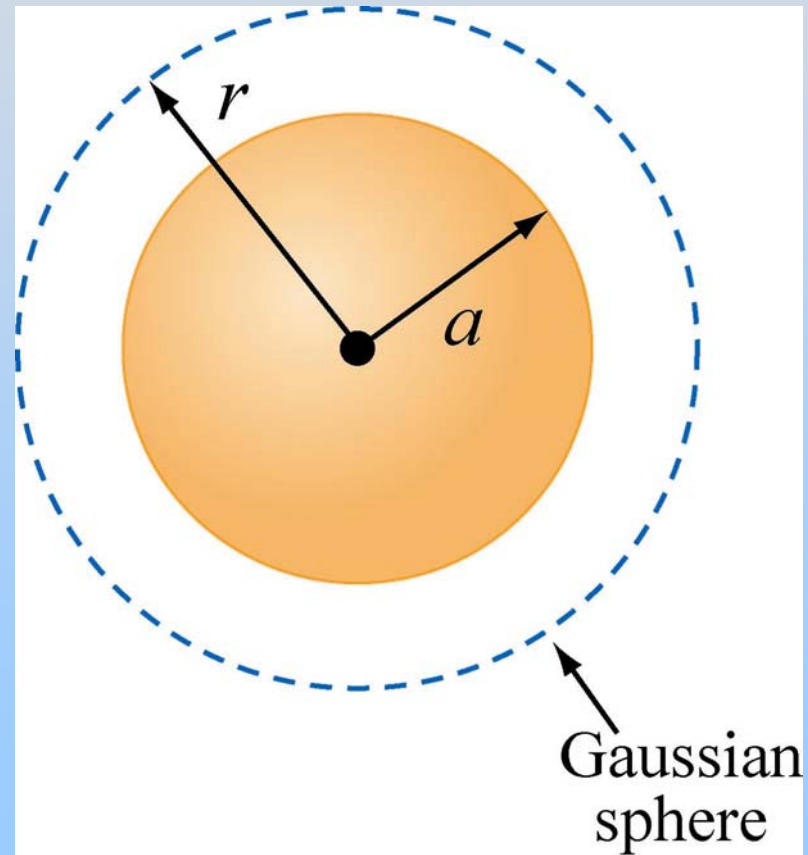
Region 1: $r > a$

Total charge enclosed $q_{in} = +Q$

$$\begin{aligned}\Phi_E &= \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA \\ &= E(4\pi r^2)\end{aligned}$$

$$\Phi_E = 4\pi r^2 E = \frac{q_{in}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \Rightarrow \vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$



Gauss: Spherical Symmetry

Region 2: $r < a$

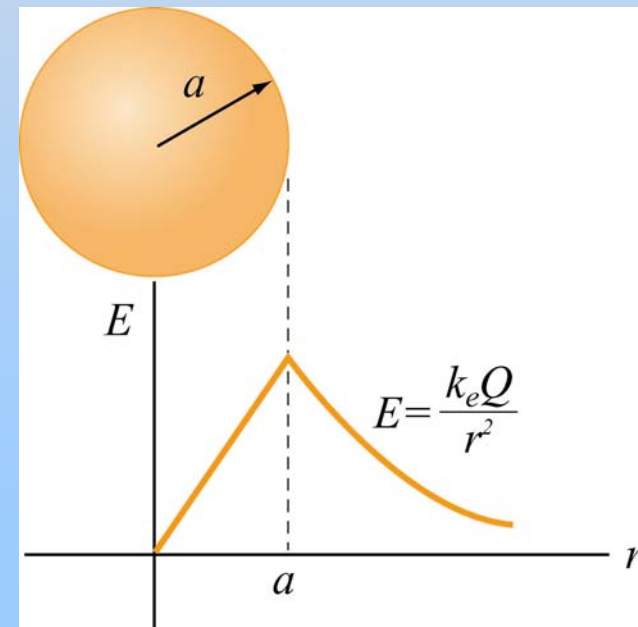
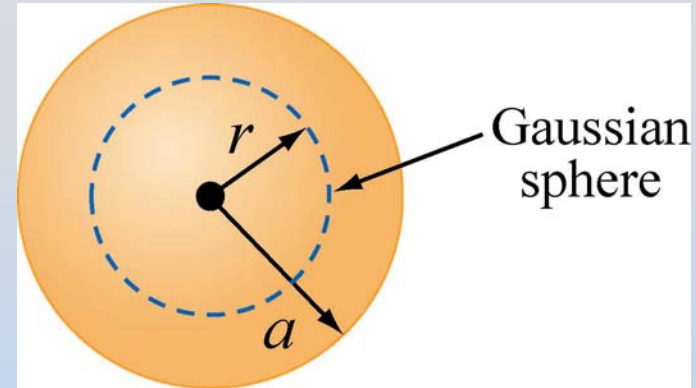
Total charge enclosed:

$$q_{in} = \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3} \right) Q = \left(\frac{r^3}{a^3} \right) Q \quad \text{OR} \quad q_{in} = \rho V$$

Gauss's law:

$$\Phi_E = E(4\pi r^2) = \frac{q_{in}}{\epsilon_0} = \left(\frac{r^3}{a^3} \right) \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0} \frac{r}{a^3} \Rightarrow \vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{r}{a^3} \hat{r}$$

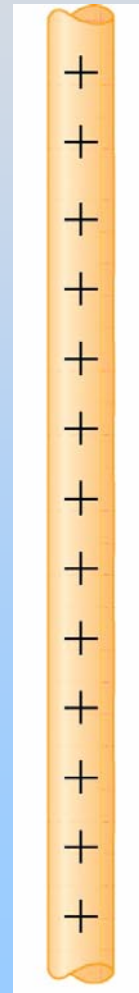


PRS Question: Field Inside Spherical Shell

Gauss: Cylindrical Symmetry

Infinitely long rod with uniform charge density λ

Find \mathbf{E} outside the rod.



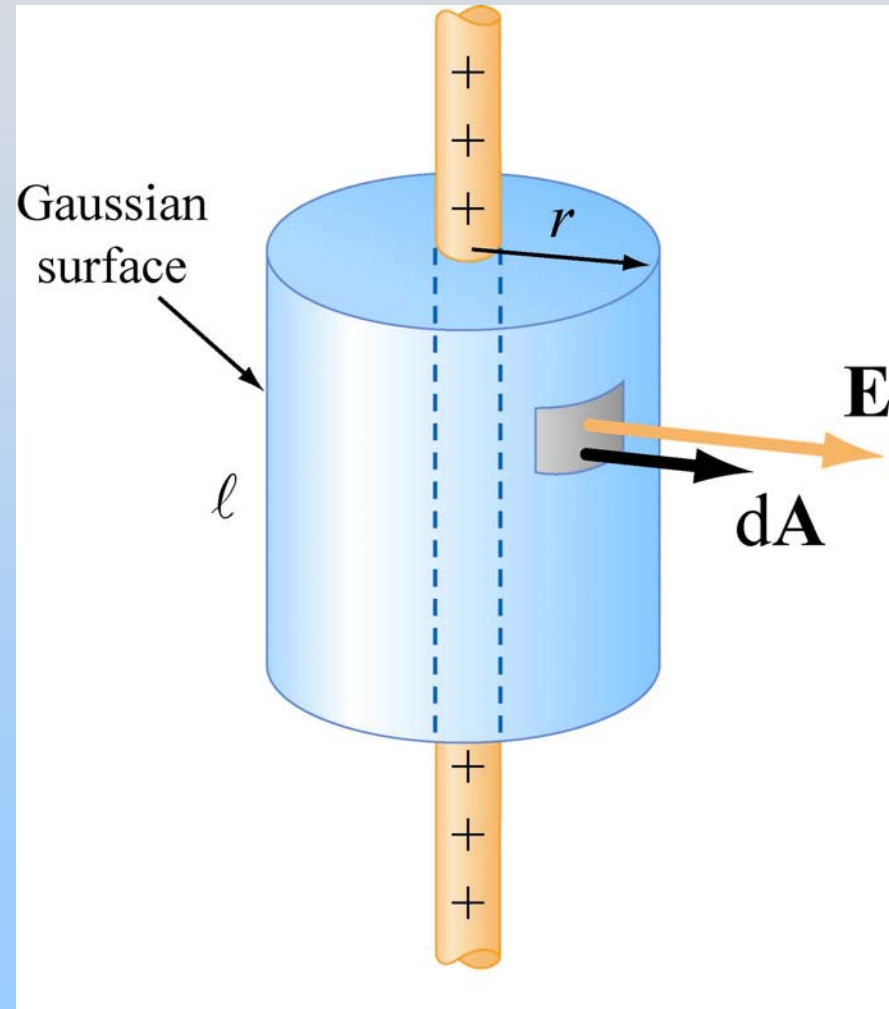
Gauss: Cylindrical Symmetry

Symmetry is Cylindrical

$$\vec{\mathbf{E}} = E \hat{\mathbf{r}}$$

Use Gaussian Cylinder

Note: r is arbitrary **but** is the radius for which you will calculate the E field!
 ℓ is arbitrary and should divide out

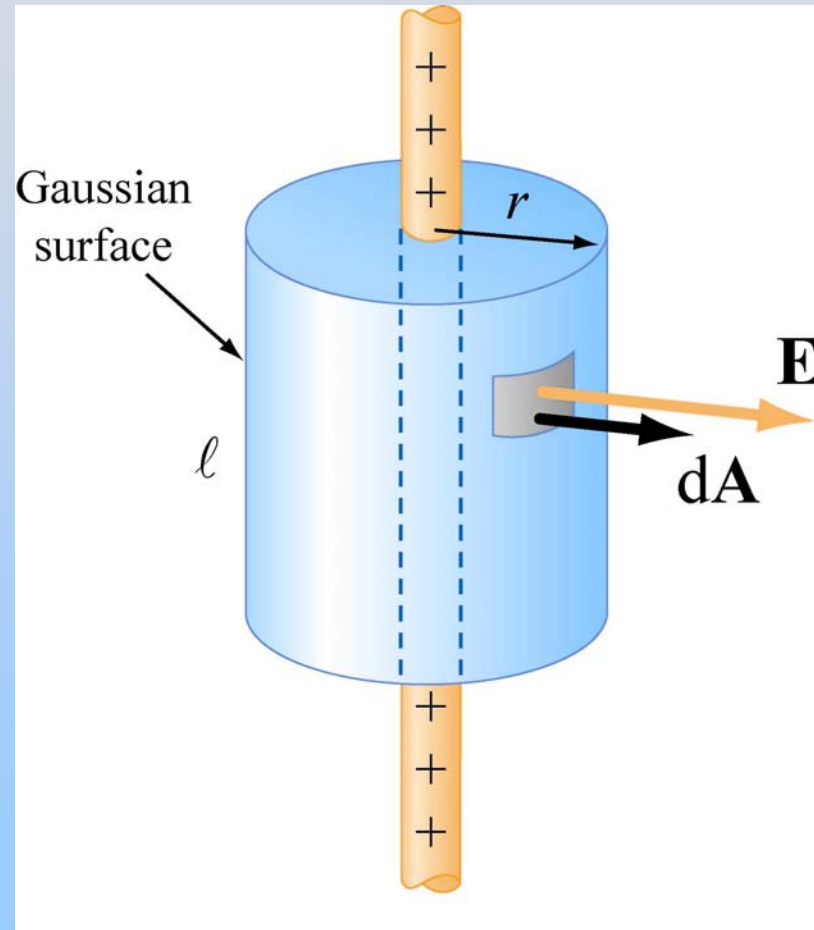


Gauss: Cylindrical Symmetry

Total charge enclosed: $q_{in} = \lambda \ell$

$$\begin{aligned}\Phi_E &= \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA \\ &= E(2\pi r \ell) = \frac{q_{in}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0}\end{aligned}$$

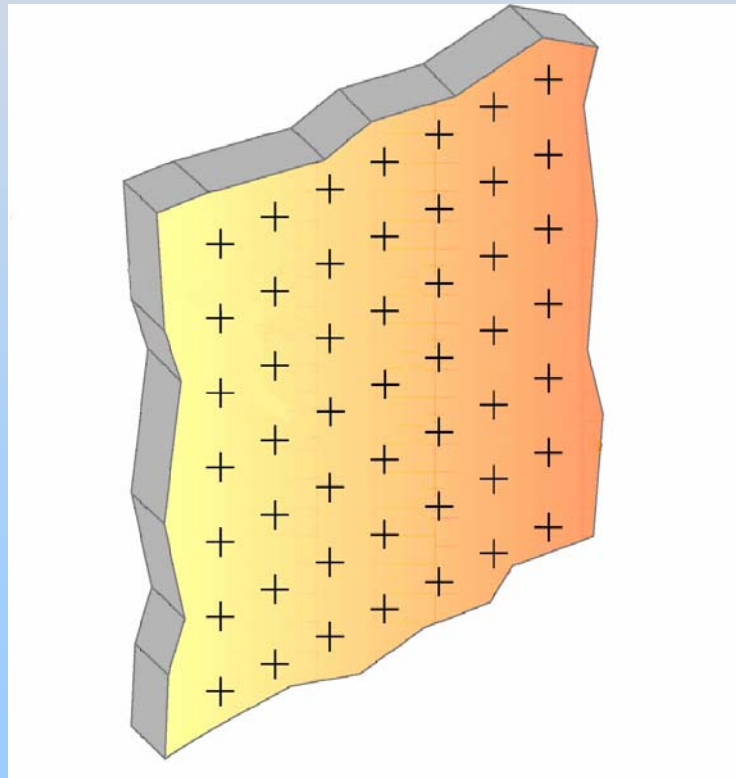
$$E = \frac{\lambda}{2\pi\epsilon_0 r} \Rightarrow \vec{\mathbf{E}} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{\mathbf{r}}$$



Gauss: Planar Symmetry

Infinite slab with uniform charge density σ

Find \mathbf{E} outside the plane



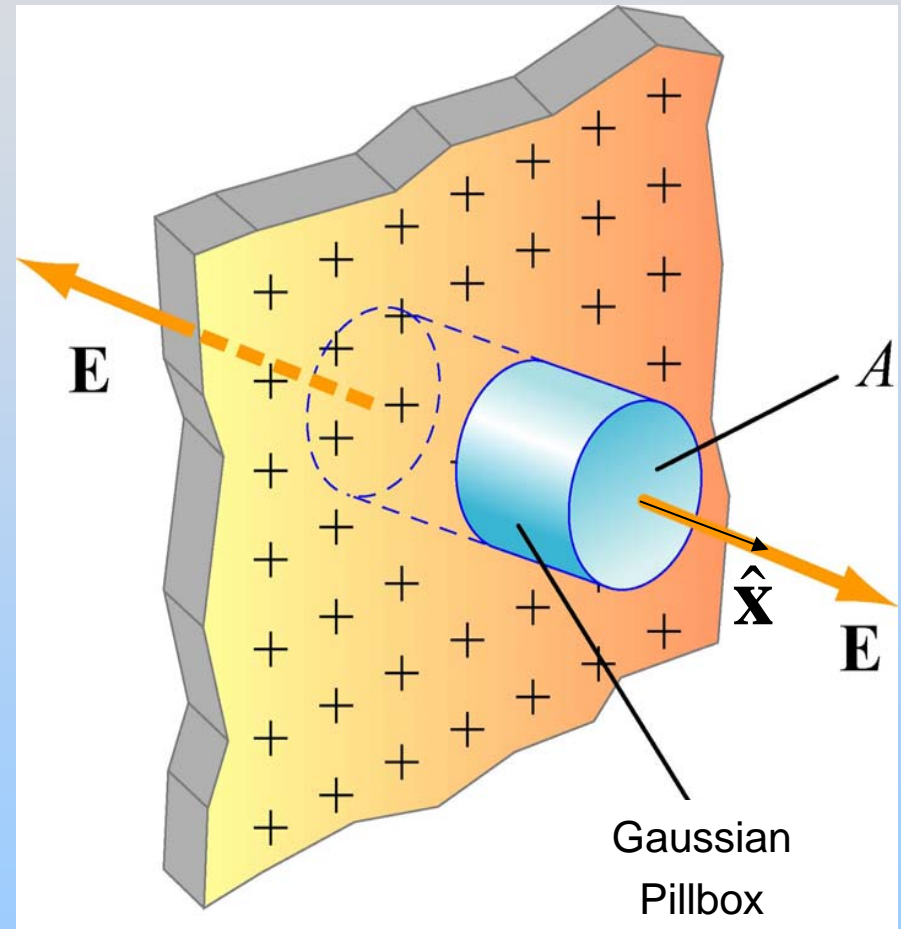
Gauss: Planar Symmetry

Symmetry is Planar

$$\vec{\mathbf{E}} = \pm E \hat{\mathbf{x}}$$

Use Gaussian Pillbox

Note: A is arbitrary (its size and shape) and should divide out



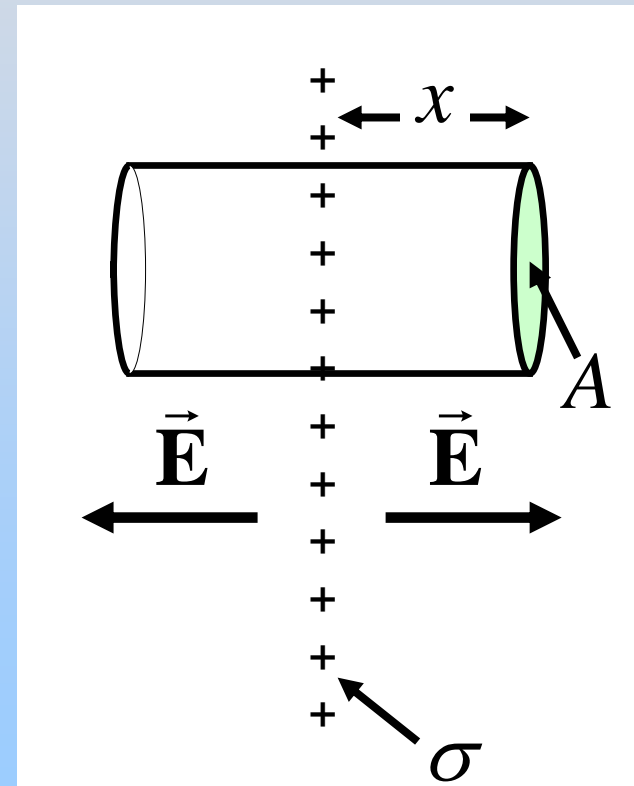
Gauss: Planar Symmetry

Total charge enclosed: $q_{in} = \sigma A$

NOTE: No flux through side of cylinder, only endcaps

$$\begin{aligned}\Phi_E &= \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oiint_S dA = EA_{\text{Endcaps}} \\ &= E(2A) = \frac{q_{in}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}\end{aligned}$$

$$E = \frac{\sigma}{2\epsilon_0} \Rightarrow \vec{\mathbf{E}} = \frac{\sigma}{2\epsilon_0} \begin{cases} \hat{\mathbf{x}} & \text{to right} \\ -\hat{\mathbf{x}} & \text{to left} \end{cases}$$



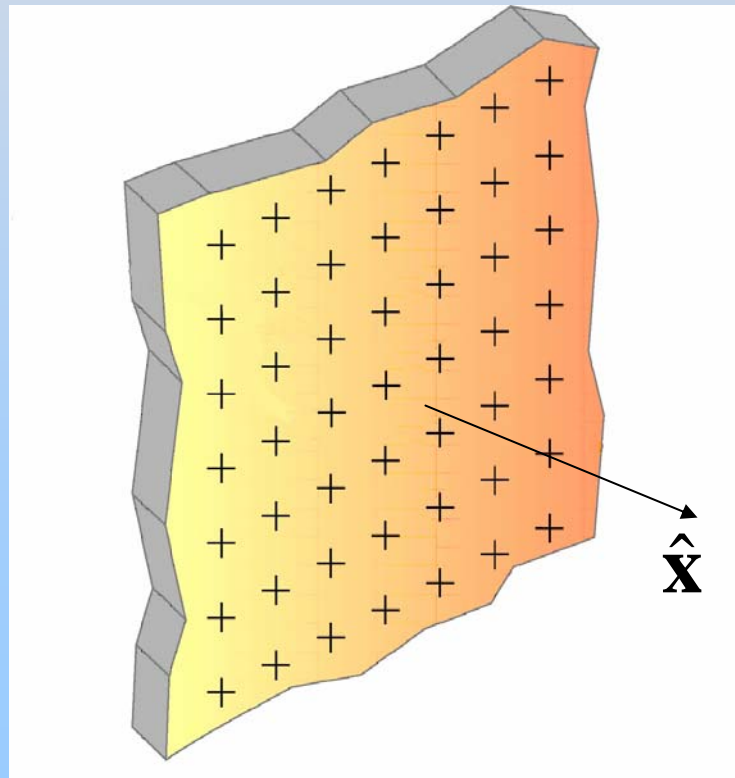
PRS Question: Slab of Charge

Group Problem: Charge Slab

Infinite slab with uniform charge density ρ

Thickness is $2d$ (from $x=-d$ to $x=d$).

Find \mathbf{E} everywhere.



PRS Question: Slab of Charge

Potential from E

Potential for Uniformly Charged Non-Conducting Solid Sphere

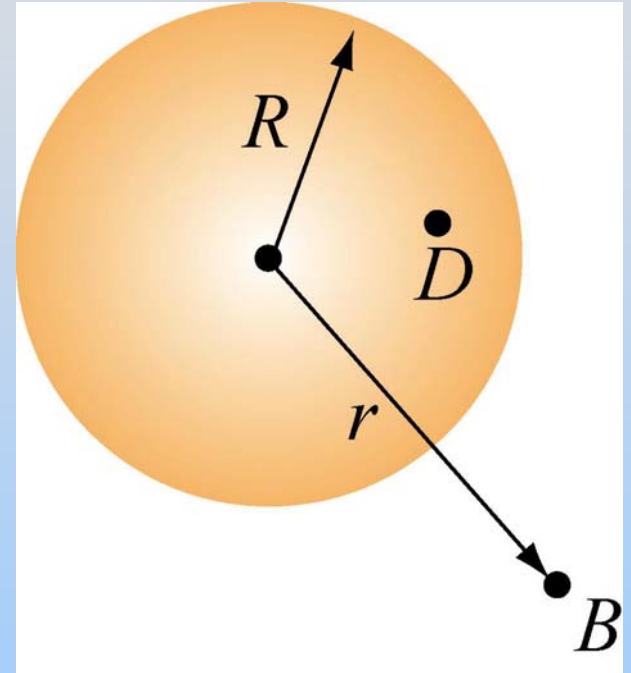
From Gauss's Law

$$\vec{\mathbf{E}} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}, & r > R \\ \frac{Qr}{4\pi\epsilon_0 R^3} \hat{\mathbf{r}}, & r < R \end{cases}$$

$$\text{Use } V_B - V_A = - \int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

Region 1: $r > a$

$$V_B - \underbrace{V(\infty)}_{=0} = - \int_{\infty}^r \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$



Point Charge!

Potential for Uniformly Charged Non-Conducting Solid Sphere

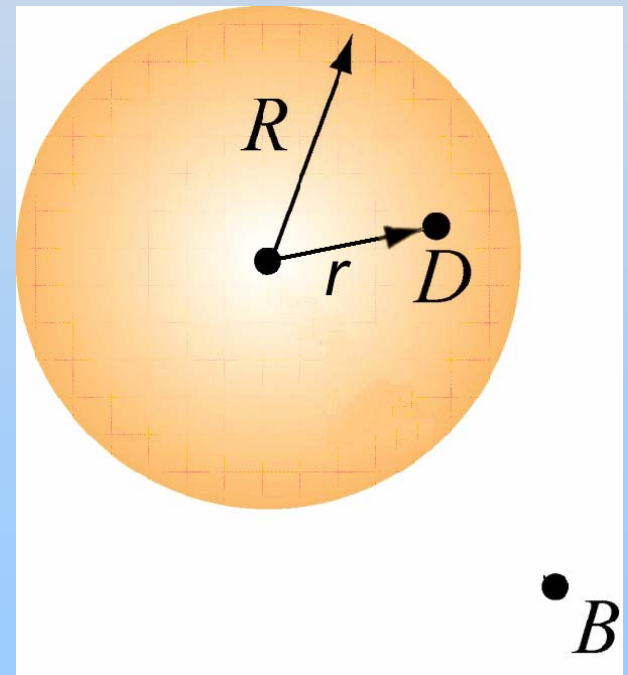
Region 2: $r < a$

$$V_D - \underbrace{V(\infty)}_{=0} = -\int_{\infty}^R dr E(r > R) - \int_R^r dr E(r < R)$$

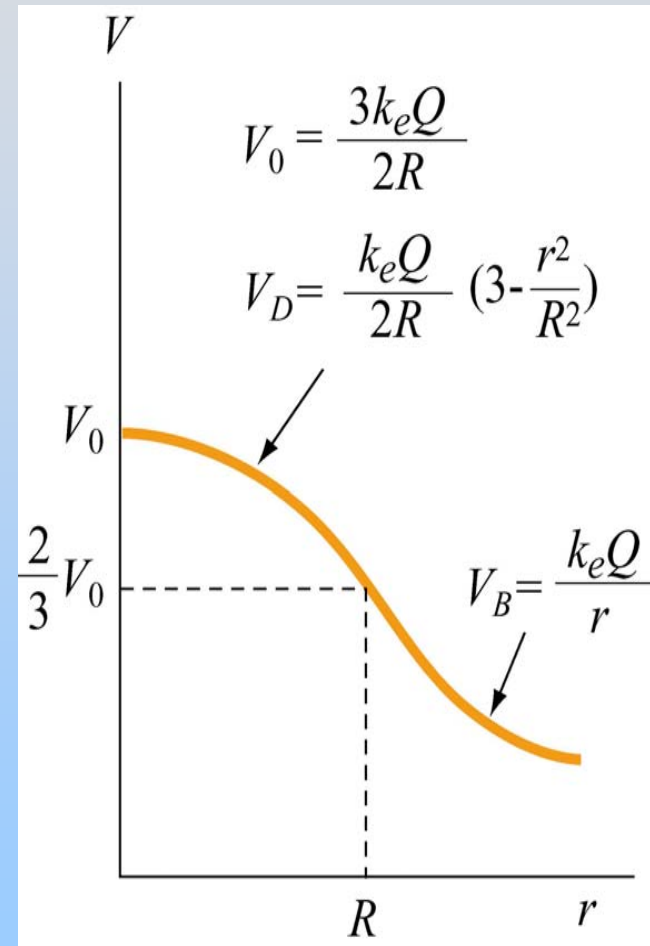
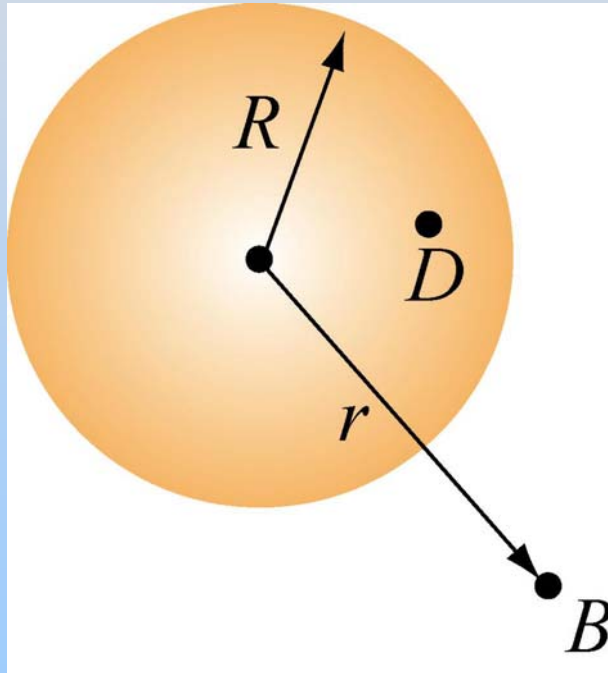
$$= -\int_{\infty}^R dr \frac{Q}{4\pi\epsilon_0 r^2} - \int_R^r dr \frac{Qr}{4\pi\epsilon_0 R^3}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{R} - \frac{1}{4\pi\epsilon_0} \frac{Q}{R^3} \frac{1}{2} (r^2 - R^2)$$

$$= \frac{1}{8\pi\epsilon_0} \frac{Q}{R} \left(3 - \frac{r^2}{R^2} \right)$$



Potential for Uniformly Charged Non-Conducting Solid Sphere

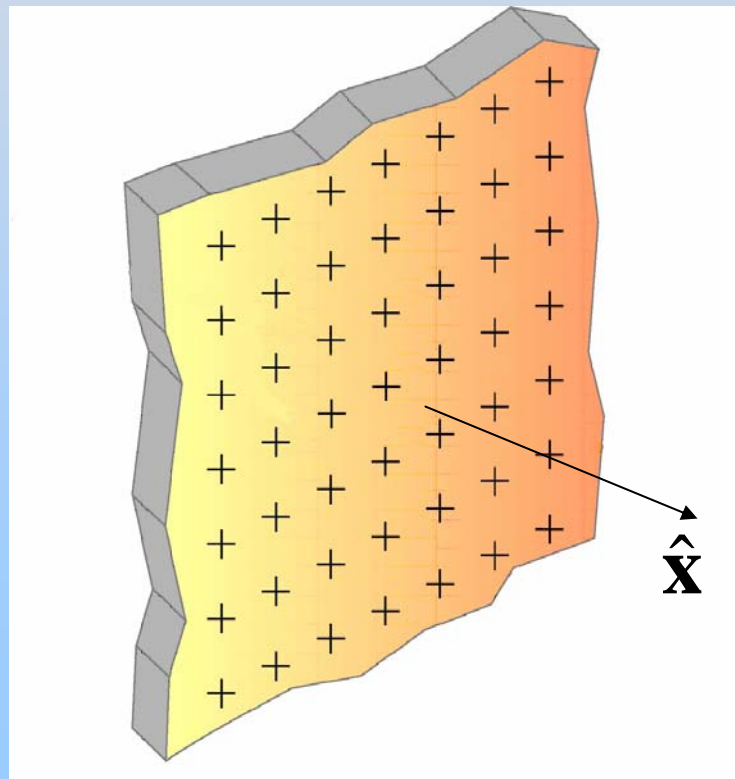


Group Problem: Charge Slab

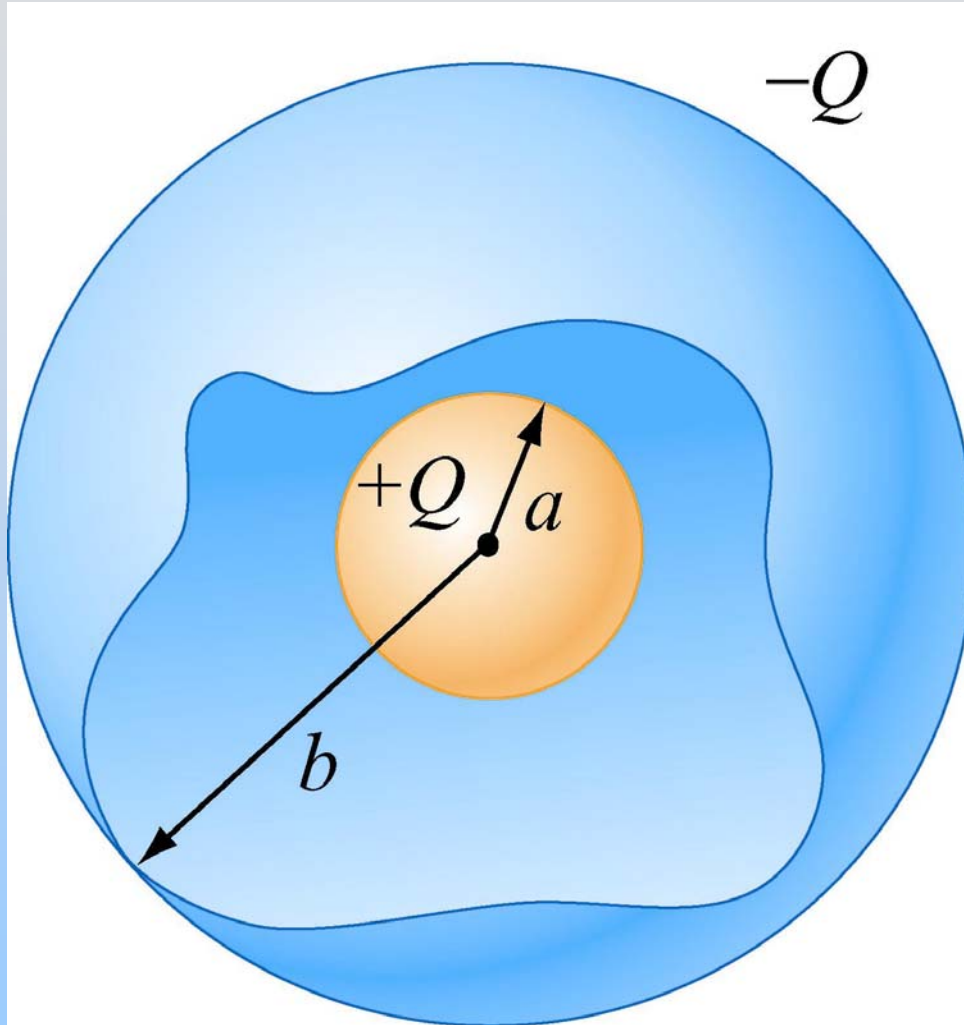
Infinite slab with uniform charge density ρ

Thickness is $2d$ (from $x=-d$ to $x=d$).

If $V=0$ at $x=0$ (definition) then what is $V(x)$ for $x>0$?



Group Problem: Spherical Shells



These two spherical shells have equal but opposite charge.

Find E everywhere

Find V everywhere
(assume $V(\infty) = 0$)