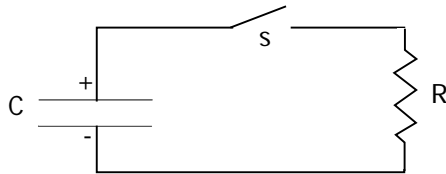


Capacitors in circuits

- A new way of looking at problems:
 - Until now: charges at rest or constant currents
 - When capacitors present: currents vary over time



- Consider the following situation:
 - A capacitor C with charge $Q_0 \rightarrow V_0 = Q_0/C$
 - A resistor R in series connected by switch s
- What happens when switch s is closed?

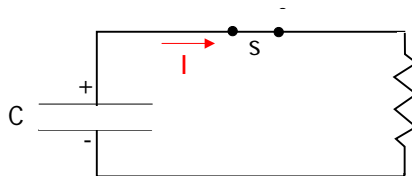
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Discharging capacitors: qualitative

- Before switch s is closed:
 - Difference in potential between C plates: V_0
 - No current circulating in the circuit (open)



- After switch s is closed:
 - Difference in potential between capacitor plates will induce current I
 - As I flows, charge difference on capacitor decreases $\rightarrow V_C$ decreases $\rightarrow I$ decreases over time

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Discharging capacitors: quantitative

- Apply second Kirchhoff's law:
 - EMF supplied by capacitor C: $V=Q/C$
 - NB: this is true at any moment in time $\rightarrow Q(t) \rightarrow V(t)$
 - Voltage drop on the resistor: $-IR$

$$\frac{Q}{C} - IR = 0$$

- Not useful in this form since $I=I(Q)$
 - $I=-dQ/dt$ (- sign because C is losing charge)

$$\frac{Q}{C} + \frac{dQ}{dt} R = 0$$

- Easy integral yields to exponential decay of the charge:

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

How to integrate RC circuits

To solve $\frac{Q}{C} + \frac{dQ}{dt} R = 0$, rewrite as: $\frac{dQ}{Q} = -\frac{dt}{RC}$

Integrate both sides:

$$\int_{Q_0}^{Q(t)} \frac{dQ}{Q} = -\int_0^t \frac{dt}{RC}$$

$$\ln \frac{Q(t)}{Q_0} = -\frac{t}{RC}$$

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

NB: $\tau=RC$ is called "decay constant" of the circuit

Solution of RC circuit

- Solution: $Q(t) = Q_0 e^{-\frac{t}{RC}}$
 - Exponential decay of charge stored in capacitor
 - $\tau = RC$ is called "decay constant" of the circuit
 - After a time RC , the charge decreased by $1/e$ w.r.t. original value
 - Units of RC :
 - cgs: $[R] = \text{statvolt s /esu}$; $[C] = \text{esu/statvolt} \rightarrow [RC] = \text{s}$
 - SI: $[R] = \text{V/A}$; $[C] = \text{C/V}$; $\text{A} = \text{C/s} \rightarrow [RC] = \text{s}$

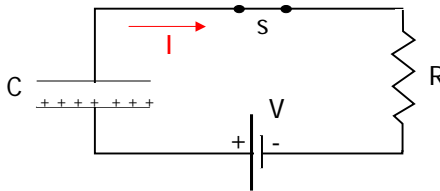
- Derive the current:

$$I(t) = -\frac{dQ}{dt} = -Q_0 \frac{d}{dt} \left(e^{-\frac{t}{RC}} \right) = \frac{Q_0}{RC} e^{-\frac{t}{RC}}$$

- Same exponential decay as for $Q(t)$

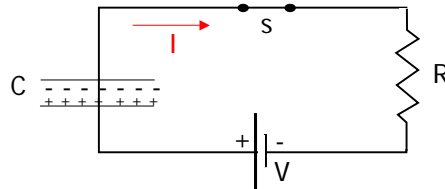
Charging capacitors

- Now 3 elements in circuit: EMF, capacitor and resistor
 - Capacitor starts uncharged



- What happens when switch s is closed?
 - When s is closed, current will suddenly flow and C will charge
 - As C charges, E opposite to EMF builds up and slows down current
 - $I(t)$ stops when V_c reaches V

Charging capacitor: solve the circuit



- Solve using Kirchoff's second law: $V - \frac{Q}{C} - IR = 0$
 - $I(t) = +dQ/dt$
 - NB: + because the capacitor is now charging!
- First order differential equation $\frac{dQ}{dt}R + \frac{Q}{C} - V = 0$
- Solution: $Q(t) = CV \left(1 - e^{-\frac{t}{RC}} \right)$

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Details of integration

To solve $\frac{dQ}{dt}R + \frac{Q}{C} - V = 0$, rewrite as: $\frac{dQ}{dt} = -\frac{(Q - CV)}{RC}$

Setting: $Q' = Q - CV$

$$\Rightarrow \frac{dQ'}{Q'} = -\frac{dt}{RC}$$

Integrating between $t=0$ and t :

$$\int_{Q=0}^{Q=Q(t)} \frac{dQ'}{Q'} = -\int_{t=0}^{t=t} \frac{dt}{RC} \Rightarrow \ln \frac{Q(t) - CV}{-CV} = -\frac{t}{RC} \Rightarrow \frac{Q(t) - CV}{CV} = -e^{-\frac{t}{RC}}$$

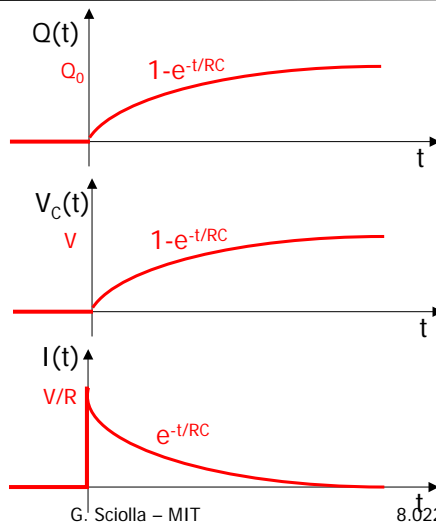
$$Q(t) = CV \left(1 - e^{-\frac{t}{RC}} \right)$$

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Graphical solution



$$Q(t) = CV \left(1 - e^{-\frac{t}{RC}} \right)$$

$$V_c(t) = Q(t)/C = V \left(1 - e^{-\frac{t}{RC}} \right)$$

$$I(t) = \frac{dQ(t)}{dt} = \frac{V}{R} e^{-\frac{t}{RC}}$$

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Important comments

- Solution of RC circuit: $V_c(t) = V \left(1 - e^{-\frac{t}{RC}} \right)$; $I(t) = \frac{V}{R} e^{-\frac{t}{RC}}$

- Are Kirchhoff's laws valid at any moment in time?

$$V - \frac{Q}{C} - IR = V - V \left(1 - e^{-\frac{t}{RC}} \right) - R \frac{V}{R} e^{-\frac{t}{RC}} = 0 \quad \text{OK!}$$

- Asymptotic behavior of the capacitor:

- At $t=0$: $I=V/R$ as if C were a short circuit
- At $t=\infty$, $I=0$ as if C were an open circuit

- Conclusion: no need to solve the differential equation!

- Solution is an exponential with time constant RC
- Asymptotic behavior of C gives initial/final values for $V(t)$ and $I(t)$

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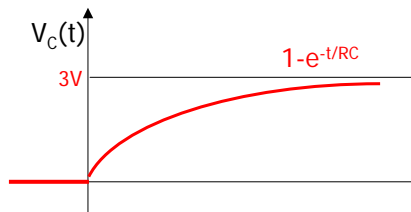
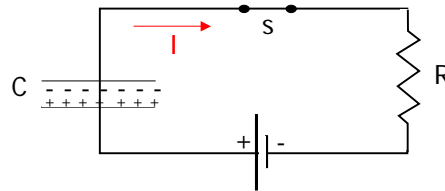
Time constant of RC circuit (E9)

- Simple RC circuit with

- $V_{EMF} = 3\text{ V}$
- $C = 1.3\text{ F}$
- $R = 11.7\ \Omega$

- Questions:

- What are V_C and I ?
- Verify that time constant is RC



$$V_C(t) = V_{EMF} \left(1 - e^{-\frac{t}{RC}} \right)$$

$$RC = 15.2\text{ s}$$

If formula is correct \Rightarrow

$$V_C = V_{EMF} (1 - 1/e) = 1.9\text{ V} \text{ when } t = 15.2$$

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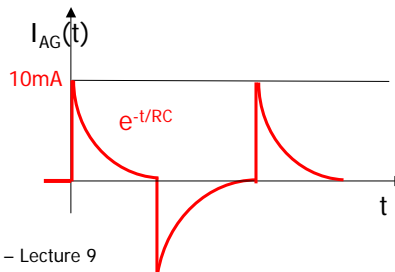
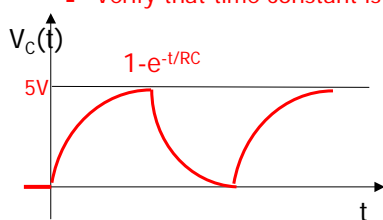
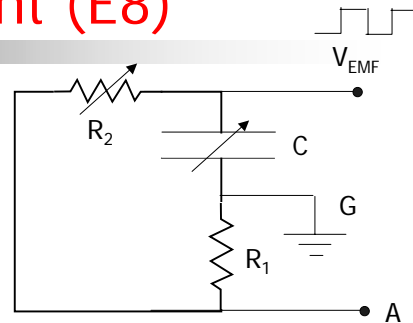
Verify time constant (E8)

- RC circuit with

- V_{EMF} = squared 5 V pulses
- Variable C initially = $0.3\ \mu\text{F}$
- Variable R_2 initially = $400\ \Omega$
- $R_1 = 100\ \Omega$

- Display on scope V_C and $I(R_1)$

- Verify that time constant is RC



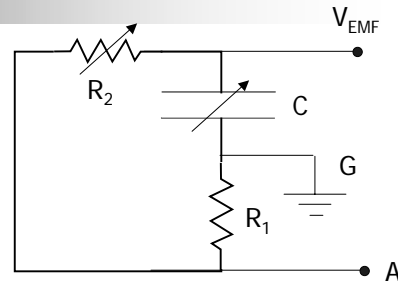
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Verify time constant (E8)

- RC circuit with
 - V_{EMF} = squared 5 V pulses
 - Variable C initially = $0.3 \mu\text{F}$
 - Variable R_2 initially = 400Ω
 - $R_1 = 100 \Omega$

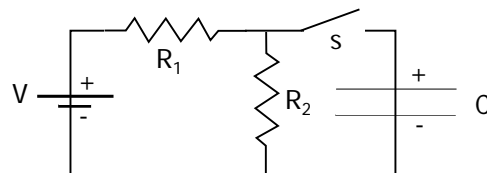


Assuming $\tau = RC \dots$

- What happens when we double C?
 - $\tau_1 = RC' = 2RC = 2\tau_0 \rightarrow V(I_{AG})$ raises (falls) twice as fast
- How should we change R_2 to have the same effect?
 - $R' = 2R = 2(R_1 + R_2) \rightarrow R_2': 400 \rightarrow 900 \Omega$

More complicated RC circuits

- What if the RC circuit is more than just a series of R and C?
- Consider the following circuit:



- Calculate $Q(t)$ on the capacitor
- Solution:
 - Kirckhoff's laws will solve it: TEDIOUS!
 - Use Thevenin's Theorem