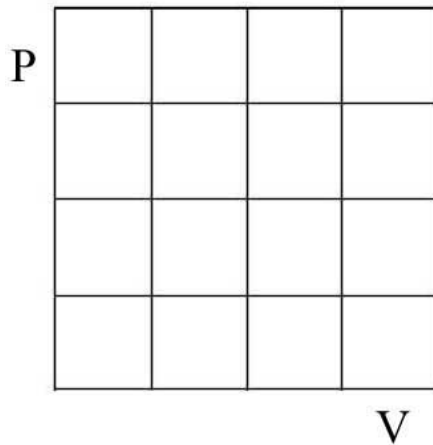


### Problem 3: Carnot Cycle of an Ideal Gas

In this problem, the starting pressure  $P_a$  and volume  $V_a$  of an ideal gas in state a, are given. The ratio  $R_V = V_c/V_a > 1$  of the volumes of the states c and a is given. Finally a constant  $\gamma = 5/3$  is given. You do not know how many moles of the gas are present.

a) Read over steps (1)- (4) below and sketch the path of the cycle on a  $P-V$  plot on the graph below. Label all appropriate points.



(1) In the first of four steps,  $a$  to  $b$ , an ideal gas is compressed from  $V_a$  to  $V_b$  while no heat is allowed to flow into or out of the system. The compression of the gas raises the temperature from an initial temperature  $T_1$  and to a final temperature  $T_2$ . During this process the quantity  $PV^\gamma = \text{constant}$ , where  $\gamma = 5/3$ .

- What is the pressure  $P_b$  and volume  $V_b$  of the state b of the gas after the compression is finished?
- What is the change in internal energy of the gas during this change of state?
- What is the work done by the gas during this compression?

(2) The gas is now allowed to expand isothermally from  $b$  to  $c$ , from volume  $V_b$  to volume  $V_c$ .

- Express the work done by the gas in this process  $W_{cb}$  and the amount of heat  $Q_{cb}$  that must be added from the heat source at  $T_2$  in terms of  $P_a$ ,  $V_a$ ,  $T_2$ ,  $T_1$ , and  $V_c$ .

Is this heat positive or negative? Explain whether it is added to the system or removed.

e) What is the pressure  $P_c$  of the gas after the expansion is finished?

(3) When the gas has reached point  $c$  it expands from  $V_c$  to  $V_d$  while no heat is allowed to flow into or out of the system. The expansion of the gas lowers the temperature and pressure from an initial temperature  $T_2$  to a final temperature  $T_1$ . During this process the quantity  $PV^\gamma = \text{constant}$ .

f) What is the pressure  $P_d$  and volume  $V_d$  of the state  $d$  of the gas after the expansion is finished?

g) What is the change in internal energy of the gas during this change of state?

h) What is the work done by the gas during this expansion?

(4) The gas is now compressed isothermally from  $d$  to  $a$  at constant  $T_1$  from volume  $V_d$  back to  $V_a$ .

i) Find the work done by the system on the surroundings  $W_{ad}$  and the amount of heat  $Q_{ad}$  that flows between the system and the surroundings. Are these quantities positive or negative? Explain whether heat is added to the system or removed from the heat source at  $T_1$ .

#### **Total Cycle:**

j) What is the total work  $W_{cycle}$  done by the gas during this cycle?

k) What is the total heat  $Q_{cycle}$  (*from*  $T_2$ ) drawn from the higher temperature heat source during this cycle?

l) What is the efficiency of this cycle  $\varepsilon_{\max} = W_{cycle} / Q_{cycle}$  (*from*  $T_2$ )?

#### **Problem 4 Heat pump**

A reversible heat engine can be run in the other direction, in which case it does negative work  $W_{cycle}$  on the world while “pumping” heat  $Q_{cycle}$  (*into*  $T_2$ ) into a reservoir at an upper temperature,  $T_2$ , from a lower temperature,  $T_1$ . The heat gain of this cycle, defined to be

$$g \equiv Q_{\text{cycle}}(\text{into } T_2) / W_{\text{cycle}} = (1 / \varepsilon_{\text{max}})$$

where  $\varepsilon_{\text{max}} = (T_2 - T_1) / T_2$  is the maximum thermodynamic efficiency of a heat engine. The refrigerator performance is defined to be

$$K \equiv Q_{\text{cycle}}(\text{from } T_1) / W_{\text{cycle}} = T_1 / (T_2 - T_1)$$

Consider that you have a large swimming pool and plan to heat your house with a heat pump that pumps heat from the pool into your house. A large plate in the water will remain at  $0^\circ\text{C}$  due to the formation of ice. You pick  $T_2$  to be  $50^\circ\text{C}$ , which will be the temperature of the (large) radiators used to heat your house. Assume that your heat pump has the maximum efficiency allowed by thermodynamics.

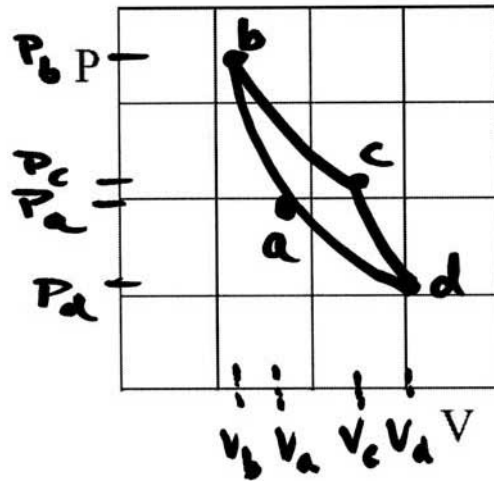
- What is the heat gain and the refrigerator performance for this cycle? Be careful to use units of Kelvin for temperature.
- If your house formerly burned 1200 gallons of oil in a winter (at \$2.00/gallon), how much will the electricity cost (at \$0.10 per kilowatt-hour) to replace this heat using the heat pump? A gallon of oil has mass 3.4 kg and contains  $1.4 \times 10^8 \text{ J} \cdot \text{gal}^{-1}$ .
- The ice cube that appears in your pool over the winter will be how many meters on each side? (It takes  $3.35 \times 10^6 \text{ J}$  to melt one kg of ice; it takes up this much heat when freezing.) The density of ice is  $0.931 \times 10^3 \text{ kg} \cdot \text{m}^{-3}$ .

This would be great for cooling your house in the summer – even if the pool warmed up enough to swim in it, you could still cool your house by running the heat pump in reverse as an air conditioner! More practically, you might be able to use ground water (and the dirt around it) as the heat sink.

### Problem 3: Carnot Cycle of an Ideal Gas

In this problem, the starting pressure  $P_a$  and volume  $V_a$  of an ideal gas in state a, are given. The ratio  $R_V = V_c/V_a > 1$  of the volumes of the states c and a is given. Finally a constant  $\gamma = 5/3$  is given. You do not know how many moles of the gas are present.

a) Read over steps (1)- (4) below and sketch the path of the cycle on a  $P-V$  plot on the graph below. Label all appropriate points.



(1) In the first of four steps,  $a$  to  $b$ , an ideal gas is compressed from  $V_a$  to  $V_b$  while no heat is allowed to flow into or out of the system. The compression of the gas raises the temperature from an initial temperature  $T_1$  and to a final temperature  $T_2$ . During this process the quantity  $PV^\gamma = \text{constant}$ , where  $\gamma = 5/3$ .

a) What is the pressure  $P_b$  and volume of the gas  $V_b$  after the compression is finished?

**Answer:** According to the ideal gas law,  $P_b V_b = n_m R T_2$  and  $P_a V_a = n_m R T_1$  so

$$P_b V_b = P_a V_a \frac{T_2}{T_1}.$$

So the pressure

$$P_b = P_a \frac{V_a}{V_b} \frac{T_2}{T_1}$$

The compression satisfies  $P_b V_b^\gamma = P_a V_a^\gamma$  so using the above result for pressure  $P_b$ , we get

$$P_b V_b^\gamma = P_a \frac{V_a T_2}{V_b T_1} V_b^\gamma = P_a V_a^\gamma.$$

This becomes using  $\gamma = 5/3$

$$V_b^{2/3} = \frac{T_1}{T_2} V_a^{2/3}$$

The volume  $V_b$  is then

$$V_b = \left( \frac{T_1}{T_2} \right)^{3/2} V_a.$$

Thus the ratio of the volumes is

$$\frac{V_a}{V_b} = \left( \frac{T_2}{T_1} \right)^{3/2}$$

So the pressure  $P_b$  is

$$P_b = P_a \left( \frac{T_2}{T_1} \right)^{5/2}$$

b) What is the change in internal energy of the gas during this change of state?

**Answer:** The change in internal energy is

$$U_b - U_a = \frac{3}{2} n_m R \Delta T = \frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$$

c) What is the work done by the gas during this compression?

**Answer:** Since no heat is exchanged  $Q_{ba} = 0$

$$U_b - U_a = -W_{ba} + Q_{ba} = -W_{ba} = \frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$$

So

$$W_{ba} = -\frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1} < 0.$$

The surroundings do work compressing the gas.

(2) The gas is now allowed to expand isothermally from  $b$  to  $c$ , from volume  $V_b$  to volume  $V_c$ .

- d) Express the work done by the gas in this process  $W_{cb}$  and the amount of heat  $Q_{cb}$  that must be added from the heat source at  $T_2$  in terms of  $P_a$ ,  $V_a$ ,  $T_2$ ,  $T_1$ , and  $V_c$ . Is this heat positive or negative? Explain whether it is added to the system or removed.

**Answer:** This is an isothermal expansion so the temperature does not change  $\Delta T = 0$ . Thus the internal energy is constant,

$$U_c - U_b = \frac{3}{2} n_m R \Delta T = 0.$$

The gas does work on the surroundings because it is expanding. The pressure is not constant during this expansion. Since the gas is expanding by an isothermal process, the Ideal Gas Law relates the pressure and volume variation according to

$$P = \frac{n_m R T}{V}.$$

Therefore the work done by the gas on the surroundings is the integral

$$W_{cb} = n_m R T_2 \int_{V_b}^{V_c} \frac{dV}{V} = n_m R T_2 \ln(V_c / V_b).$$

Using the result for the volume  $V_b$  from part a)

$$V_b = \left( \frac{T_1}{T_2} \right)^{3/2} V_a,$$

the work is

$$W_{cb} = n_m R T_2 \int_{V_b}^{V_c} \frac{dV}{V} = n_m R T_2 \ln \left( \left( \frac{T_2}{T_1} \right)^{3/2} V_c / V_a \right)$$

Recall that the volumes are related according to  $R_V = V_c / V_a > 0$  and  $n_m R = P_a V_a / T_1$  so the work done is positive and given by

$$W_{cb} = n_m R T_2 \ln(V_c / V_a) = P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V) > 0$$

From The First Law of Thermodynamics,

$$0 = U_c - U_b = -W_{cb} + Q_{cb},$$

Thus the heat that flows into the system from the heat source at temperature  $T_2$  is equal to the work done by the expanding gas.

$$Q_{cb} = W_{cb} = P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V) > 0,$$

Note that this heat flow must flow from the higher temperature heat source into the system because as the gas expands it should lose internal energy and would decrease its temperature unless heat flows into the system keeping the internal energy and hence the temperature constant.

e) What is the pressure  $P_c$  of the gas after the expansion is finished?

**Answer:**  $P_c V_c = n_m R T_2 = \frac{P_a V_a}{T_1} T_2$ . Thus

$$P_c = \frac{P_a V_a T_2}{V_c T_1} = \frac{P_a T_2}{R_V T_1}.$$

(3) When the gas has reached point  $c$  it expands from  $V_c$  to  $V_d$  while no heat is allowed to flow into or out of the system. The expansion of the gas lowers the temperature and pressure from an initial temperature  $T_2$  to a final temperature  $T_1$ . During this process the quantity  $PV^\gamma = \text{constant}$ .

f) What is the pressure  $P_d$  and the volume  $V_d$  of the state  $d$  of the gas after the expansion is finished?

**Answer:** This calculation is identical to part a), with state  $d$  replacing state  $a$ , and state  $c$  replacing state  $b$ . So the volume  $V_b$  is then

$$V_c = \left(\frac{T_1}{T_2}\right)^{3/2} V_d.$$

Thus the ratio of the volumes is

$$\frac{V_d}{V_c} = \left(\frac{T_2}{T_1}\right)^{3/2}$$

So the pressure  $P_c$  is

$$P_c = P_d \left(\frac{T_2}{T_1}\right)^{5/2}$$

hence

$$P_d = P_c \left(\frac{T_1}{T_2}\right)^{5/2}$$

g) What is the change in internal energy of the gas during this change of state?

**Answer:** The decrease in the internal energy is due to the temperature decrease of the ideal gas during expansion

$$U_d - U_c = \frac{3}{2} P_a V_a \frac{(T_1 - T_2)}{T_1}$$

h) What is the work done by the gas during this expansion?

**Answer:** Since no heat is exchanged  $Q_{dc} = 0$

$$U_d - U_c = -W_{dc} + Q_{dc} = -W_{dc} = \frac{3}{2} P_a V_a \frac{(T_1 - T_2)}{T_1}$$

So

$$W_{dc} = \frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1} > 0.$$

The gas does work on the surroundings since the gas is expanding.

(4) The gas is now compressed isothermally from  $d$  to  $a$  at constant  $T_1$  from volume  $V_d$  back to  $V_a$ .

i) Find the work done by the system on the surroundings  $W_{ad}$  and the amount of heat  $Q_{ad}$  that flows between the system and the surroundings. Are these quantities



positive or negative? Explain whether heat is added to the system or removed from the heat source at  $T_1$ .

**Answer:** When the gas undergoes compression it will increase its internal energy but heat flows out of the system maintaining constant internal energy,  $\Delta U = 0$  and hence the compression is isothermal. The calculation of the work and heat is similar to step (2) except the temperature is held at  $T_1$ . The work done by the system on the surroundings is negative and is given by the integral

$$W_{ad} = n_m RT_1 \int_{V_d}^{V_a} \frac{dV}{V} = n_m RT_1 \ln(V_a / V_d) = P_a V_a \ln(V_a / V_d) = -P_a V_a \ln(R_V).$$

From part f) the volume  $V_d = \left(\frac{T_2}{T_1}\right)^{3/2} V_c$  so the work done is

$$W_{ad} = n_m RT_1 \int_{V_d}^{V_a} \frac{dV}{V} = n_m RT_1 \ln(V_a / V_d) = P_a V_a \ln(V_a / \left(\frac{T_2}{T_1}\right)^{3/2} V_c) = -P_a V_a \ln\left(\left(\frac{T_2}{T_1}\right)^{3/2} R_V\right)$$

According to the First Law this is equal to the heat that flows into the system which is also negative which means that it actually flows out of the system into the surroundings at temperature  $T_1$ ,

$$Q_{ad} = W_{ad} = -P_a V_a \ln\left(\left(\frac{T_2}{T_1}\right)^{3/2} R_V\right).$$

**Total Cycle:**

j) What is the total work  $W_{cycle}$  done by the gas during this cycle?

**Answer:** The work done by the heat engine on the surroundings during the cycle is positive and given by

$$W_{cycle} = P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V - P_a V_a \ln\left(\left(\frac{T_2}{T_1}\right)^{3/2} R_V\right) = P_a V_a \ln\left(\left(\frac{T_2}{T_1}\right)^{3/2} R_V\right) \left(\frac{T_2}{T_1} - 1\right).$$

k) What is the total heat  $Q_{cycle}$  (from  $T_2$ ) drawn from the higher temperature heat source during this cycle?

**Answer:** The heat that flowed from the higher temperature heat source  $T_2$  occurred during step (2)  $b \rightarrow c$  isothermal expansion,

$$Q^{total} \text{ taken from heat source at } T_2 = P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V).$$

1) What is the efficiency of this cycle  $\varepsilon_{\max} = W_{\text{cycle}} / Q_{\text{cycle}} (\text{from } T_2)$ ?

**Answer: The efficiency is given by ratio of the work done divided by the heat flowing into the system from the higher temperature heat source**

$$\varepsilon_{\max} = W_{\text{cycle}} / Q_{\text{cycle}} (\text{from } T_2) = P_a V_a \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V \left(\frac{T_2}{T_1} - 1\right) / P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$$

$$\varepsilon_{\max} = \left(\frac{T_2}{T_1} - 1\right) / \frac{T_2}{T_1} = \frac{T_2 - T_1}{T_2} = \frac{\Delta T}{T_2}.$$

Table 1: Summary of Heat Engine

Process	$U_f - U_i$	$W_{f,i}$	$Q_{f,i}$
$a \rightarrow b$ adiabatic compression	$\frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$ positive	$-\frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$ negative	0
$b \rightarrow c$ isothermal expansion	0	$P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ positive	$P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ positive from $T_2$
$c \rightarrow d$ adiabatic expansion	$\frac{3}{2} P_a V_a \frac{(T_1 - T_2)}{T_1}$ negative	$\frac{3}{2} P_a V_a \frac{(T_2 - T_1)}{T_1}$ positive	0
$d \rightarrow a$ isothermal compression	0	$-P_a V_a \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ negative	$-P_a V_a \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ negative, into $T_1$
Total	0	$P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ $-P_a V_a \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ positive	$P_a V_a \frac{T_2}{T_1} \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ $-P_a V_a \ln\left(\frac{T_2}{T_1}\right)^{3/2} R_V)$ positive, from $T_2$ into $T_1$