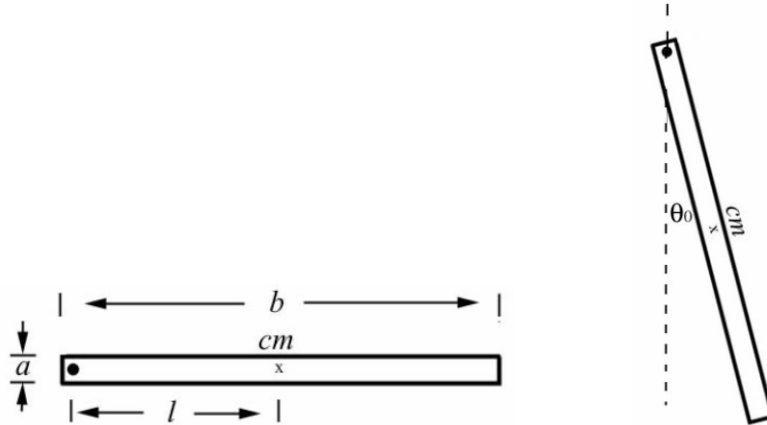


## Problem 2: Experiment 09 Physical Pendulum

### Part One: Ruler Pendulum

The ruler has a mass  $m_r = 0.159$  kg, a width  $a = 0.028$  m, a length  $b = 1.00$  m, and the distance from the pivot point to the center of mass is  $l = 0.479$  m.



Enter your measured period into the  $T_{\text{meas}}$  column of the table below and calculate the other entries using the formulas

$$T_{\text{ideal}} = 2\pi\sqrt{l/g}$$

and

$$T_{\text{theory}} = 2\pi\sqrt{l/g} \sqrt{\frac{I}{ml^2} \left(1 + \frac{\theta_0^2}{16}\right)}$$

with  $g = 9.805 \text{ ms}^{-2}$ .

$\theta_0$	$T_{\text{meas}}$	$T_{\text{theory}}$	$T_{\text{ideal}}$
0.10			
0.52			

### Part Two: Added Mass

Consider the effect of a brass weight clipped to the ruler. The weight is shaped like a washer with an outer radius  $r_o = 0.016$  m and an inner radius  $r_i = 0.002$  m; it has a mass  $m_w = 0.050$  kg. It is clipped to the ruler so that the inner hole is over the 0.500 m mark on the ruler, or  $l = 0.479$  m from the pivot point. The clip has a mass  $m_c = 0.0086$  kg and you may assume its center of mass is also over the 0.500 m mark on the ruler. If you treat the washer and clip as point masses,

then, as was discussed in the notes for Experiment 08, the combined unit (ruler, weight and clip) has a moment of inertia about the pivot point

$$I_p = \frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2$$

where  $d = l$  for this situation. The restoring torque that tries to return the pendulum to a vertical position will be

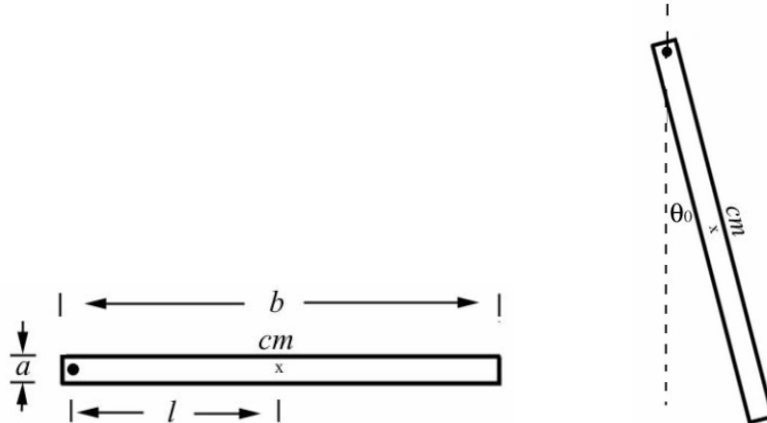
$$\tau = (m_r l + m_c d + m_w d)g \sin \theta \approx (m_r l + m_c d + m_w d)g \theta$$

1. Use these two expressions to derive an equation of motion for the pendulum and calculate its period  $T$  in the small amplitude ( $\sin \theta \cong \theta$ ) approximation. Express your answer algebraically in terms of the variables  $a$ ,  $b$ ,  $d$ ,  $l$ ,  $m_r$ ,  $m_w$ ,  $m_c$ , and  $g$ .
2. Evaluate your result numerically and compare with the value you measured in your experiment.
3. If you treat the brass object washer as a point mass, its moment of inertia about the pivot point  $P$  is  $I_{w,P} = m_w l^2$ . If the brass object is a washer with an inner radius  $r_i$  and outer radius  $r_o$ , then moment of inertia about its center of mass given by  $I_w = \frac{1}{2} m_w (r_o^2 + r_i^2)$ . If the washer is a solid disc with radius  $r$ , the moment of inertia about its center of mass given by  $I_w = \frac{1}{2} m_w r^2$ . When this is taken into account, what is the new (and more accurate) expression for  $I_{w,P}$ ? How many percent does this differ from the simpler expression  $I_{w,P} = m_w l^2$ ?

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### Part One: Ruler Pendulum

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Enter your measured period into the  $T_{\text{meas}}$  column of the table below and calculate the other entries using the formulas

$$T_{\text{ideal}} = 2\pi\sqrt{l/g}$$

and

$$T_{\text{theory}} = 2\pi\sqrt{l/g} \sqrt{\frac{I}{ml^2} \left(1 + \frac{\theta_0^2}{16}\right)}$$

with  $g = 9.805 \text{ ms}^{-2}$ .

### Solution:

The first part of the analysis of the experiment is to calculate the moment of inertia about an axis passing through the center of mass, perpendicular to the plane formed by the sides of the ruler. In particular, choose Cartesian coordinates with the origin at the center of mass, and the x-axis along the length, and y-axis along the width. The mass per unit area

$$\sigma = \frac{\text{mass}}{\text{Area}} = \frac{m_1}{ab}$$

The mass element is rotating about the z-axis in a circular orbit with radius  $r_{\perp} = (x^2 + y^2)^{1/2}$ , so the moment of inertia about the center of mass of the ruler is

$$I_{cm} = \int_{body} (r_{\perp,cm})^2 dm = \sigma \int_{x=-b/2}^{x=b/2} \left( \int_{y=-a/2}^{y=a/2} (x^2 + y^2) dy \right) dx$$

We first do the integral in the y-direction,

$$I_{cm} = \frac{m_1}{ab} \int_{x=-b/2}^{x=b/2} \left( \left( x^2 y + \frac{y^3}{3} \right) \Big|_{y=-a/2}^{y=a/2} \right) dx = \frac{m_1}{ab} \int_{x=-b/2}^{x=b/2} \left( x^2 a + \frac{a^3}{12} \right) dx$$

We now do the integral in the x-direction

$$I_s = \frac{m_1}{ab} \left( \frac{x^3}{3} a + \frac{a^3}{12} x \right) \Big|_{x=-b/2}^{x=b/2} = \frac{m_1}{ab} \left( \frac{b^3}{12} a + \frac{a^3}{12} b \right) = \frac{m_1}{12} (b^2 + a^2)$$

$$I_s = \frac{(0.159 \text{ kg})}{12} \left( (1.000 \text{ m})^2 + (0.028 \text{ m})^2 \right) = 1.326 \times 10^{-2} \text{ kg} \cdot \text{m}^2$$

Now use the parallel axis theorem to calculate the moment of inertia about the pivot point,

$$I_s = \frac{m_1}{12} (b^2 + a^2) + m_1 l_{cm}^2.$$

Using the data for the ruler, the moment about the pivot point is

$$I_s = 1.326 \times 10^{-2} \text{ kg} \cdot \text{m}^2 + (0.159 \text{ kg})(0.479 \text{ m})^2 = 4.97 \times 10^{-2} \text{ kg} \cdot \text{m}^2$$

$\theta_0$	$T_{\text{meas}}$	$T_{\text{theory}}$	$T_{\text{ideal}}$
0.10	1.622	1.623	1.390
0.25	1.633	1.629	1.390
0.45	1.642	1.643	1.390

## Part Two: Added Mass

Consider the effect of a brass weight clipped to the ruler. The weight is shaped like a washer with an outer radius  $r_o = 0.016$  m and an inner radius  $r_i = 0.002$  m; it has a mass  $m_w = 0.050$  kg. It is clipped to the ruler so that the inner hole is over the 0.500 m mark on the ruler, or  $l = 0.479$  m from the pivot point. The clip has a mass  $m_c = 0.0086$  kg and you may assume its center of mass is also over the 0.500 m mark on the ruler. If you treat the washer and clip as point masses,

then, as was discussed in the notes for Experiment 08, the combined unit (ruler, weight and clip) has a moment of inertia about the pivot point

$$I_p = \frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2$$

where  $d = l$  for this situation. The restoring torque that tries to return the pendulum to a vertical position will be

$$\tau = (m_r l + m_c d + m_w d)g \sin \theta \approx (m_r l + m_c d + m_w d)g \theta$$

1. Use these two expressions to derive an equation of motion for the pendulum and calculate its period  $T$  in the small amplitude ( $\sin \theta \approx \theta$ ) approximation. Express your answer algebraically in terms of the variables  $a$ ,  $b$ ,  $d$ ,  $l$ ,  $m_r$ ,  $m_w$ ,  $m_c$ , and  $g$ .

**Answer:**

a) The equation of motion for the pendulum is

$$\tau = I_p \frac{d^2 \theta}{dt^2}$$

or, when we substitute expressions for  $\tau$  and  $I_p$ ,

$$-(m_r l + m_c d + m_w d)g \sin \theta = \left( \frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2 \right)$$

For small amplitudes  $\sin \theta \approx \theta$  and the equation becomes the familiar equation  $-kx = mdx^2/dt^2$ , so we immediately find the angular frequency

$$\omega = \sqrt{\frac{g(m_r l + m_c d + m_w d)}{\frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2}}$$

The period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2}{g(m_r l + m_c d + m_w d)}}$$

2. Evaluate your result numerically and compare with the value you measured in your experiment.

**Solution**

The moment of inertia of the washer and binder clip treated as point masses about the pivot point is

$$I_{S,c,w} = (m_c + m_w)l_{cm}^2 = 0.0586 \text{ kg} (0.479 \text{ m})^2 = 1.345 \times 10^{-2} \text{ kg} \cdot \text{m}^2$$

Thus the total moment of inertia is

$$I_S^{total} = \frac{m}{12}(b^2 + a^2) + ml_{cm}^2 + (m_c + m_w)l_{cm}^2$$

$$I_S^{total} = 4.97 \times 10^{-2} \text{ kg} \cdot \text{m}^2 + 1.345 \times 10^{-2} \text{ kg} \cdot \text{m}^2 = 6.32 \times 10^{-2} \text{ kg} \cdot \text{m}^2.$$

So the new period is

$$T_0 = \frac{2\pi}{\omega_p} \cong 2\pi \sqrt{\frac{I_S^{total}}{l_{cm} m^{total} g}} = 2\pi \sqrt{\frac{6.32 \times 10^{-2} \text{ kg} \cdot \text{m}^2}{(0.479 \text{ m})(0.218 \text{ kg})(9.805 \text{ m} \cdot \text{s}^{-2})}} = 1.56 \text{ s}$$

So the approximation gives good agreement with the data.

Displacement [m]	Measured Period [s]	Calculated Period $T_0$ [s]
0.20	1.585	1.56

3. If you treat the brass object washer as a point mass, its moment of inertia about the pivot point  $P$  is  $I_{w,P} = m_w l^2$ . If the brass object is a washer with an inner radius  $r_i$  and outer radius  $r_o$ , then moment of inertia about its center of mass given by  $I_w = \frac{1}{2} m_w (r_o^2 + r_i^2)$ . If the washer is a solid disc with radius  $r$ , the moment of inertia about its center of mass given by  $I_w = \frac{1}{2} m_w r^2$ . When this is taken into account, what is the new (and more accurate) expression for  $I_{w,P}$ ? How many percent does this differ from the simpler expression  $I_{w,P} = m_w l^2$ ?

**Solution:**

In Problem 1, you showed that the moment of inertia of a washer about the center of mass is given by the result,  $I_{cm} = \frac{1}{2} m_w (r_o^2 + r_i^2)$ , so the total moment of inertia is now

$$I_S^{total} = \frac{m}{12}(b^2 + a^2) + ml_{cm}^2 + (m_c + m_w)l_{cm}^2 + \frac{1}{2} m_w (r_o^2 + r_i^2)$$

The moment of inertia of the washer about the center of mass is

$$I_{cm,w} = \frac{1}{2} m_w (r_0^2 + r_i^2) = \frac{1}{2} (0.050 \text{ kg}) \left( (0.016 \text{ m})^2 + (0.002 \text{ m})^2 \right) = 6.5 \times 10^{-6} \text{ kg} \cdot \text{m}^2.$$

This is negligible compared to the overall moment of inertia. The ratio of the moment of the washer about its center of mass compared to the moment of the washer treated as a point mass about the pivot point is

$$\frac{I_{cm,w}}{m_w l_{cm}^2} = \frac{6.5 \times 10^{-6} \text{ kg} \cdot \text{m}^2}{1.147 \times 10^{-2} \text{ kg} \cdot \text{m}^2} = 5.7 \times 10^{-4}$$