

Rotation and Translation of Rigid Body

- Torque produces angular acceleration about center of mass

$$\sum_i^{\text{torques}} \tau_{cm,i} = I_{cm} \alpha_{cm}$$

- I_{cm} is the moment of inertial about the center of mass
- α is the angular acceleration about center of mass or any other point in a rigid body.
- This is really a vector relation; only z-component is non-zero if problem is planar

Fixed Axis Rotation: Kinematics

Angle variable

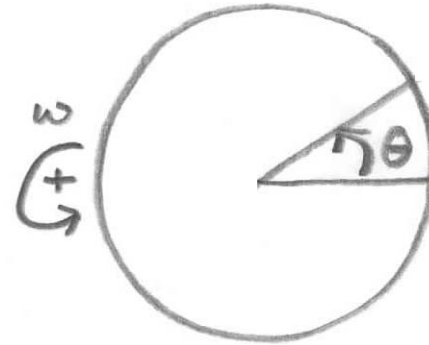
$$\theta$$

Angular velocity

$$\omega \equiv \frac{d\theta}{dt}$$

Angular acceleration

$$\alpha \equiv \frac{d^2\theta}{dt^2}$$



These are exactly analogous to the variables x , v_x , and a_x for One dimensional motion, and obey the same equations

For constant angular acceleration:

$$\theta(t) = \theta_0 + \omega_0(t - t_0) + \frac{1}{2}\alpha(t - t_0)^2$$

$$\omega(t) = \omega_0 + \alpha(t - t_0)$$

$$[\omega(t)]^2 = \omega_0^2 + 2\alpha(\theta(t) - \theta_0)$$

Fixed Axis Rotation: Tangential Velocity and Tangential Acceleration

Kinematics of individual mass elements:

$$\Delta m_i$$

- Tangential velocity

$$v_{\text{tan},i} = r_{\perp,i} \omega$$

- Tangential acceleration

$$a_{\text{tan},i} = r_{\perp,i} \alpha$$

- Radial Acceleration

$$a_{\text{rad},i} = \frac{v_{\text{tan},i}^2}{r_{\perp,i}} = r_{\perp,i} \omega^2$$

PRS: Ladybug Acceleration

A ladybug sits at the outer edge of a merry-go-round that is turning and slowing down. The tangential component of the ladybug's (Cartesian) acceleration is

QuickTime™ and a
Graphics decompressor
are needed to see this picture.

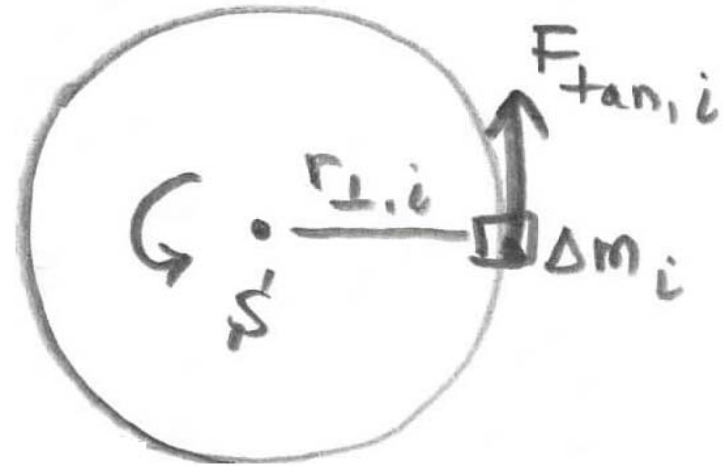
1. in the +x direction.
2. in the -x direction.
3. in the +y direction.
4. in the -y direction.
5. in the +z direction.
6. in the -z direction.
7. zero.

Torque

Torque about axis S:

$$\tau_{S,i} = \mathbf{r}_{S,i} \times \mathbf{F}_i$$

- Counterclockwise +z direction
- perpendicular to the plane



$$\tau_{S,i} = r_{S\perp,i} F_{\tan,i} = r_{S\perp,i} \Delta m_i a_y = \Delta m_i (r_{S\perp,i})^2 \alpha$$

torques

$$\sum_i \tau_{S,i} = I_S \alpha_{cm} = I_S \alpha_S$$

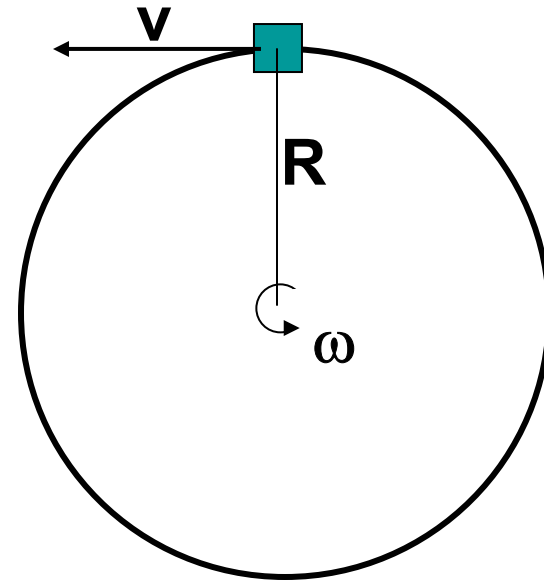
Energy of Rotating Mass

The mass is rotating:

Angular velocity ω

Radius R

Speed v



Speed $v = \omega R$

Kinetic Energy = $\frac{1}{2} m v^2 = \frac{1}{2} m R^2 \omega^2 = \frac{1}{2} I_s \omega^2$

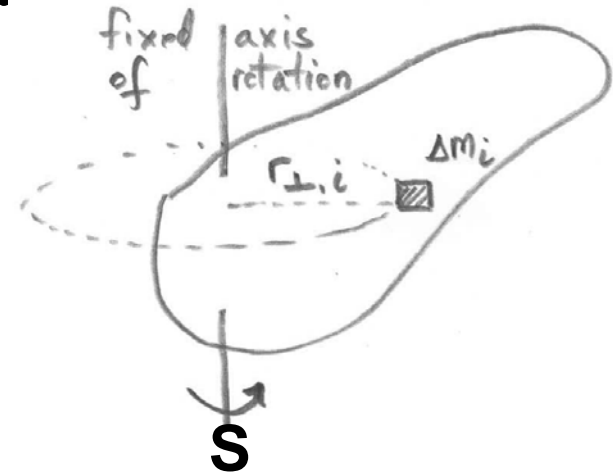
$$K_S^{rot} = \frac{1}{2} I_S \omega_S^2$$

Moment of Inertia - Idea

Mass element Δm_i

Radius of orbit $r_{\perp,i}$

Moment of Inertia about S $I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{S\perp,i})^2$



The moment of inertia takes the place of mass in the Dynamical ($F=ma$) and Energy ($K=mv^2/2$) equations for rotational motion.

$$\sum_i^{\text{torques}} \tau_{S,i} = I_S \alpha_{cm} \qquad K_S^{\text{rot}} = \frac{1}{2} I_S \omega_S^2$$

These work about ANY NON-ACCELERATING axis

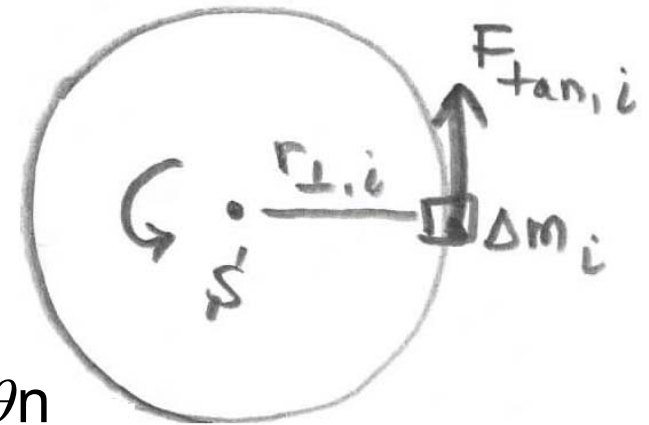
Rotational Work

Starting from

$$\Delta W_i = \mathbf{F}_i \cdot \Delta \mathbf{r}_i$$

$$\mathbf{F}_{\text{tan},i} = F_{\text{tan},i} \hat{\theta}$$

- tangential force



$$\Delta \mathbf{r}_{S,i} = (r_{S,\perp})_i \Delta \theta \mathbf{n}$$

- displacement vector

- infinitesimal work $\Delta W_i = \mathbf{F}_i \cdot \Delta \mathbf{r}_i = F_{\text{tan},i} (r_{S,\perp})_i \Delta \theta = \tau_i \Delta \theta$

- Rotational work:

$$W_{fi}^{\text{rot}} = \int_i^f \tau(\theta) d\theta = \tau_{\text{avg}} (\theta_f - \theta_i)$$

Note: if τ is constant, it equals τ_{avg}

General Work-Kinetic Energy Rel'n

- Fixed axis passing through the c of m of the body

$$W_{f0}^{rot} = \frac{1}{2} I_{cm} \omega_{cm,f}^2 - \frac{1}{2} I_{cm} \omega_{cm,0}^2 = K_{rot,f} - K_{rot,0} \equiv \Delta K_{rot}$$

- Rotation and translation combined - General Motion

$$K_f = \left(\frac{1}{2} m v_{cm,f}^2 + \frac{1}{2} I_{cm} \omega_f^2 \right) = \left(\frac{1}{2} m v_{cm,0}^2 + \frac{1}{2} I_{cm} \omega_0^2 \right) + W_{fi}^{trans} + W_{fi}^{rot} = +K_0 + W_{fi}^{total}$$

$W_{total} = \Delta K_{trans} + \Delta K_{rot}$

$$W_{fi}^{rot} = \int_i^f \tau(\theta) d\theta = \tau_{avg} (\theta_f - \theta_i)$$

Note: if τ is constant, it equals τ_{avg}

Strategy: Moment of Inertia

Always start from a tabulated I_{cm} plus the Parallel Axis Theorem

Use
$$I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{S\perp,i})^2 = \int dm (r_{S\perp,dm})^2$$

When all else fails

Note:
$$I_{cm} = \int dm (r_{cm\perp,dm})^2 = \alpha MR^2$$

Where α is between 0 and 1

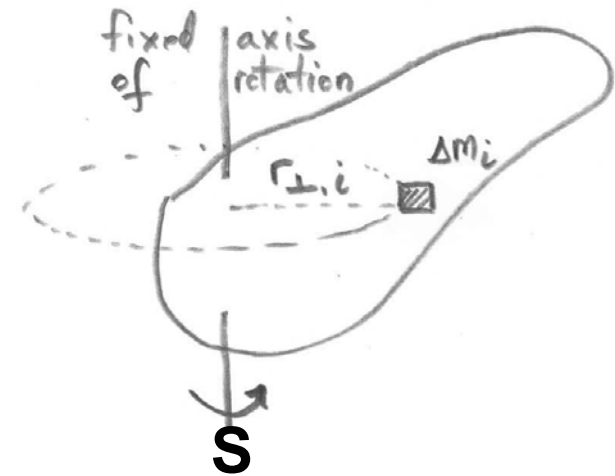


Table of I_{cm}

Object	α
Hoop	1.0
Disk	0.5
Sphere	0.4
Rod	$ML^2/12$

Rotational Dynamics & Energy - Summary

- Dynamical Equations about axis **S**

Dynamics

$$\sum_i^{\text{torques}} \tau_{S,i} = I_S \alpha_{cm} = I_S \alpha_S$$

Is not true if **S** accelerates

- Most General Equations

$$\sum_i^{\text{torques}} \tau_{cm,i} = I_{cm} \alpha_{cm}$$

$$\sum_i \vec{F}_i^{\text{ext}} = M \vec{a}_{cm}$$

OK if c of m accelerates

Kinetic Energy

$$K_S^{\text{rot}} = \frac{1}{2} I_S \omega_S^2$$

Requires **S** to be stationary

Includes KE of c of m

$$K^{\text{tot}} = \frac{1}{2} I_{cm} \omega_{cm}^2 + \frac{1}{2} M v_{cm}^2$$

KE of rotation+KE of translation

PRS - kinetic energy

A disk with mass M and radius R is spinning with angular velocity ω about an axis that passes through the rim of the disk perpendicular to its plane. Its total kinetic energy is:

1. $\frac{1}{4} M R^2 \omega^2$

4. $\frac{1}{4} M R \omega^2$

2. $\frac{1}{2} M R^2 \omega^2$

5. $\frac{1}{2} M R \omega^2$

3. $\frac{3}{4} M R^2 \omega^2$

6. $\frac{1}{2} M R \omega$

Dynamics: Translational & Rotational

Translational Dynamics

- Total Force

$$\sum \dot{\mathbf{F}}_i^{ext}$$

- Momentum of System

$$\dot{\mathbf{p}}_{system}$$

- Dynamics of Translation

$$\sum_i \mathbf{F}_i^{ext} = \frac{d\mathbf{p}_{system}^r}{dt}$$

Rotational Dynamics of point mass about S

- Torque

$$\dot{\boldsymbol{\tau}}_{S,i} = \mathbf{r}_{i,S} \times \dot{\mathbf{F}}_i$$

- Angular momentum

$$\dot{\mathbf{L}}_{S,i} = \mathbf{r}_{i,S} \times \dot{\mathbf{p}}_i$$

- Dynamics of rotation

$$\sum_i \dot{\boldsymbol{\tau}}_{S,i}^{ext} = \frac{d\dot{\mathbf{L}}_S}{dt}$$

Rotational Angular Momentum & Energy

- Putting it Together - Spin plus Translation

Angular Momentum

$$\mathbf{L}_S^{system} = I_{cm} \boldsymbol{\omega} + \sum_{i=1}^{i=N} \mathbf{R}_{cm,S} \times \mathbf{p}$$

L of rotation + L of translation

Kinetic Energy

$$K^{tot} = \frac{1}{2} I_{cm} \omega_{cm}^2 + \frac{1}{2} M v_{cm}^2$$

KE of rotation + KE of translation

Dynamics (i.e. change)

$$\sum_i \mathbf{r}_{S,i}^{ext} = \frac{d\mathbf{L}_S^{system}}{dt}$$

$$K_f = K_0 + W_{fi}^{trans} + W_{fi}^{rot}$$

cf.
$$\sum_i \mathbf{F}_i^{ext} = \frac{d\mathbf{p}}{dt}$$

Physical Content of

$$\sum_i \tau_{S,z}^{ext} = \frac{d\mathbf{L}_z^{system}}{dt}$$

- In the case that $I_{S,z}$ does not change:

$$\sum_i \tau_{S,z}^{ext} = \frac{d\mathbf{L}_z^{system}}{dt} = \frac{d(I_{S,z}\omega_z)}{dt} = I_{S,z} \frac{d\omega_z}{dt} = I_{S,z} \frac{d^2\theta}{dt^2} = I_{S,z} \alpha_z$$

- But $I_{S,z}$ may change:
 - Spinning Skater pulls in arms
 - Rain falling on merry-go-round
- Conservation of L_z is richer than p_z

$$\sum_i \tau_{S,z}^{ext} = 0 \quad \Rightarrow \quad \mathbf{L}_{z,f}^{system} = \mathbf{L}_{z,i}^{system}$$

PRS - angular momentum

A disk with mass M and radius R is spinning with angular velocity ω about an axis that passes through the rim of the disk perpendicular to its plane. The magnitude of its angular momentum is:

1. $\frac{1}{4} M R^2 \omega^2$

4. $\frac{1}{2} M R \omega$

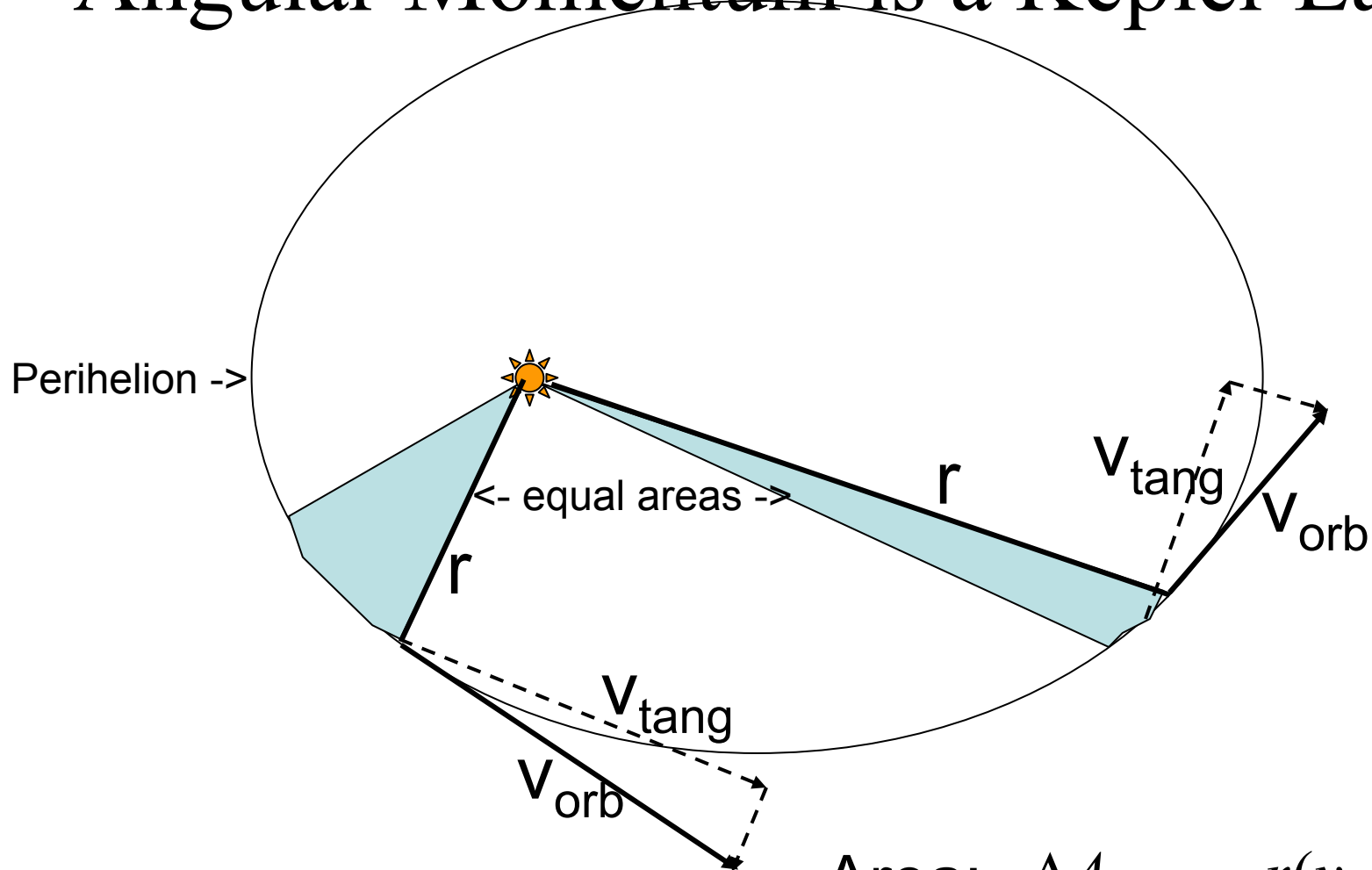
2. $\frac{1}{2} M R^2 \omega^2$

5. $\frac{3}{4} M R \omega$

3. $\frac{3}{4} M R^2 \omega^2$

6. $\frac{3}{2} M R \omega$

Angular Momentum is a Kepler Law



Angular Momentum:

$$L_{sun} = \mu v_{tang} r$$

Area: $\Delta A_{swept} = r(v_{tang} \Delta t) / 2$

$$\frac{\Delta A_{swept}}{\Delta t} = \frac{1}{2} r v_{tang} = \frac{L}{2\mu}$$

Circular Orbital Mechanics

-class problem

A planet of mass m_1 is in a circular orbit of radius R around a sun of mass m_2 .

- a. Find the period, T
- b. Find the ratio of kinetic to potential energy
- c. As R increases, all of the physical properties of the orbit (e.g. velocity) decrease except one; find it.