Name: _____ **Problem 4**: _____ /25

Problem 4(25 points) Let a be a positive real number.

(a)(3 points) Determine the range of x on which $2ax - x^2$ is nonnegative.

(b)(22 points) For that range, compute the antiderivative,

$$\int \sqrt{2ax - x^2}dx.$$

(b)(5 points) Use your answer from (a) to compute the antiderivative of

$$\int \frac{x+3}{x^2-2x+1}dx.$$

Solution to (b) By the **Solution to (a)**, the integral is,

$$\int \frac{x+3}{x^2-2x+1}dx = \int \frac{4}{(x-1)^2} + \frac{1}{x-1}dx.$$

The second integral is easily computed and gives,

$$\int \frac{x+3}{x^2-2x+1}dx = \boxed{-4/(x-1) + \ln(|x-1|) + C.}$$

Problem 4(25 points) Let a be a positive real number.

(a)(3 points) Determine the range of x on which $2ax - x^2$ is nonnegative.

Solution to (a) Completing the square gives,

$$-x^2 + 2ax = -(x-a)^2 + a^2.$$

Therefore the expression is nonnegative when,

$$-(x-a)^2 + a^2 \geq 0 \Leftrightarrow a^2 \geq (x-a)^2.$$

Taking square roots, the expression is nonnegative when,

$$a \geq x - a \geq -a.$$

This simplifies to,

$$\boxed{0 \leq x \leq 2a.}$$

(b)(22 points) For that range, compute the antiderivative,

$$\int \sqrt{2ax - x^2}dx.$$

Solution to (b) Begin by making the linear change of variables,

$$u = x - a, \quad du = dx.$$

The new integral is,

$$\int \sqrt{a^2 - u^2}du.$$

This integral can be computed using an inverse trigonometric substitution,

$$u = a\sin(\theta), \quad du = a\cos(\theta)d\theta.$$

Plugging in, the new integral is,

$$\int \sqrt{a^2(1-\sin^2(\theta))}(a\cos(\theta)d\theta) = a^2 \int \cos^2(\theta)d\theta.$$

This can be solved either using integration by parts or using the half-angle formula from trigonometry. The half-angle formula is decidedly faster and gives,

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}.$$

Substituting this in, the new integral is,

$$a^2 \int \frac{1}{2} + \frac{1}{2}\cos(2\theta)d\theta = a^2\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right) + C.$$

Using the double-angle formula, this simplifies to,

$$a^2\left(\frac{\theta}{2} + \frac{1}{2}\sin(\theta)\cos(\theta)\right) + C.$$

Back-substituting for u gives,

$$\frac{a^2}{2}\sin^{-1}(u/a) + \frac{1}{2}u\sqrt{a^2-u^2} + C.$$

Back-substituting for x gives,

$$\int \sqrt{2ax-x^2}dx = \boxed{(a^2\sin^{-1}(u/a) + (x-a)\sqrt{2ax-x^2})/2 + C.}$$

Problem 5(10 points) Compute the following derivatives. Please show all work to receive full credit.

(a)(5 points)

$$y = \sinh(t) = \frac{e^t - e^{-t}}{2}, \quad \frac{dy}{dx} = ?$$

Solution to (a) The derivative should be with respect to t. Using the chain rule, it is,

$$\frac{d}{dt}\left(\frac{e^t}{2} - \frac{e^{-t}}{2}\right) = \left(\frac{1}{2}e^t - \frac{1}{2}(-e^{-t})\right) = \boxed{\cosh(t).}$$