18.01 Exam 3

Problem 1. (20 pts) Evaluate the following integrals

a) $\int_0^2 \frac{x\,dx}{(1+x^2)^2}$

b) $\int_{-\pi/2}^{\pi/2} \sin^6 x \cos x\,dx$

Problem 2. (20 pts.) Find the following approximations to

$$\int_0^{\pi/2} \cos x\,dx$$

(Do not give a numerical approximation to square roots; leave them alone.)

a) Using the upper Riemann sum with two intervals

b) Using the trapezoidal rule with two intervals

c) Using Simpson's rule with two intervals

Problem 3. (20 points) Find the volume of the solid of revolution formed by revolving the *y*-axis the region enclosed by

$$y=\cos(x^2)$$

and the *x*-axis (central hump, only).

Problem 4. (20 points) Students studying for an exam get *x* hours of sleep in the two days leading up to the exam, where *x* is the range $0 \le x \le a$. The numbers of students who got between x_1 and x_2 hours of sleep in given by

$$\int_{x_1}^{x_2} cx\,dx, 0 \le x_1 \le x_2 \le a$$

a) What fraction o the student got less than a/2 hours of sleep?

b) Their scores are proportional to the amount of sleep they got: $S(x) = 100\,(x/a)$. Find the (correctly weighted) average score in the class.

Problem 1. (20 points) Evaluate the following integrals

10 pts a) $\int_0^2 \frac{x\,dx}{(1+x^2)^2}$ $= \frac{1}{2}\int_1^5 \frac{du}{u^2} = -\frac{1}{2u}\Big|_1^5$

$u = 1+x^2$

$du = 2x\,dx$ $= -\frac{1}{10} - \left(-\frac{1}{2}\right) =$

$= \frac{1}{2} - \frac{1}{10} = \boxed{2/5}$

10 pts b) $\int_{-\pi/2}^{\pi/2} \sin^6 x \cos x\,dx$ $= \int_{-1}^{1} u^6\,du = \frac{1}{7}u^7\Big|_{-1}^{1}$

$u = \sin x$

$du = \cos x\,dx$ $= \boxed{\frac{2}{7}}$

Problem 2. (20 points) Find the following approximations to

$$\int_0^{\pi/2} \cos x\,dx$$

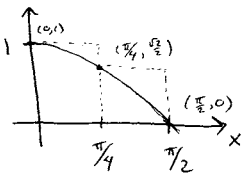
(Do not give a numerical approximation to square roots; leave them alone.)

a) using the upper Riemann sum with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cos\frac{\pi}{4}$

$= \frac{\pi}{4}\left(1+\frac{\sqrt{2}}{2}\right)$

(≈ 1.341)

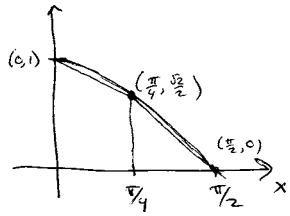


b) using the trapezoidal rule with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{1}{2}\frac{\pi}{4}\left(1+\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}$

$= \frac{\pi}{8}\left(1+\sqrt{2}\right)$

(≈ 0.948)



c) using Simpson's rule with two intervals

$\int_0^{\pi/2} \cos x\,dx \approx \frac{1}{3}\frac{\pi}{4}\left(1+4\cos\frac{\pi}{4}+0\right)$

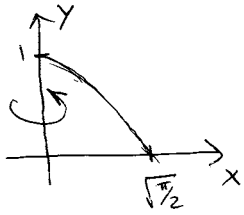
$= \frac{\pi}{12}\left(1+2\sqrt{2}\right)$

(≈ 1.002)

Problem 3. (20 points) Find the volume of the solid of revolution formed by revolving around the y-axis the region enclosed by

$$y = \cos(x^2)$$

and the x-axis (central hump, only).



$V = \int_0^{\sqrt{\pi/2}} 2\pi x \cdot \cos(x^2)\,dx$

$u = x^2$

$du = 2x\,dx$ $= \pi\int_0^{\pi/2} \cos u\,du$

$= \pi \sin u\Big|_0^{\pi/2} = \boxed{\pi}$

Problem 4. (20 points) Students studying for an exam get x hours of sleep in the two days leading up to the exam, where x is in the range $0 \le x \le a$. The number of students who got between x_1 and x_2 hours of sleep is given by

$$\int_{x_1}^{x_2} cx\,dx; \quad 0 \le x_1 \le x_2 \le a$$

10 pts a) What fraction of the students got less than $a/2$ hours of sleep?

Total number of students $= \int_0^a cx\,dx = \frac{ca^2}{2}$ (all students got between 0 and a hours of sleep).

number of students who got between 0 and $\frac{a}{2}$ hours of sleep $= \int_0^{a/2} cx\,dx = c\frac{a^2}{8}$

ratio $= \frac{ca^2/8}{ca^2/2} = \boxed{\frac{1}{4}}$

10 pts b) Their scores are proportional to the amount of sleep they got: $S(x) = 100(x/a)$. Find the (correctly weighted) average score in the class.

N = Total number of students $= \int_0^a cx\,dx = \frac{ca^2}{2}$

Average score $= \frac{1}{N}\int_0^a cx\,S(x)\,dx = \frac{1}{N}\int_0^a \frac{100c}{a}x^2\,dx$

$= \frac{1}{ca^2/2}\,\frac{100c}{a}\,\frac{a^3}{3} = \boxed{\frac{200}{3}}$ $\left(\approx 66.\overline{6}\right)$