

18.01 Practice Questions for Exam 3 – Fall 2006

1. Evaluate a) $\int_0^1 \frac{x \, dx}{\sqrt{1+3x^2}}$ b) $\int_{\pi/3}^{\pi/2} \cos^3 x \sin 2x \, dx$

2. Evaluate $\int_0^1 x \, dx$ directly from its definition as the limit of a sum.

Use upper sums (circumscribed rectangles). You can use the formula $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$.

3. A bank gives interest at the rate r , compounded continuously, so that an amount A_0 deposited grows after t years to an amount $A(t) = A_0 e^{rt}$.

You make a daily deposit at the constant annual rate k ; in other words, over the time period Δt you deposit $k\Delta t$ dollars. Set up a definite integral (give reasoning) which tells how much is in your account at the end of one year. (Do not evaluate the integral.)

4. Consider the function defined by $F(x) = \int_0^x \sqrt{3 + \sin t} \, dt$. Without attempting to find an explicit formula for $F(x)$,

a) (5) show that $F(1) \leq 2$;

b) (5) determine whether $F(x)$ is convex (“concave up”) or concave (“concave down”) on the interval $0 < x < 1$; show work or give reasoning;

c) (10) give in terms of values of $F(x)$ the value of $\int_1^2 \sqrt{3 + \sin 2t} \, dt$.

5. If $\int_0^x f(t) \, dt = e^{2x} \cos x + c$, find the value of the constant c and the function $f(t)$.

6. A glass vase has the shape of the solid obtained by rotating about the y -axis the area in the first quadrant lying over the x -interval $[0, a]$ and under the graph of $y = \sqrt{x}$. By slicing it horizontally, determine how much glass it contains.

7. A right circular cone has height 5 and base radius 1; it is over filled with ice cream, in the usual way. Place the cone so its vertex is at the origin, and its axis lies along the positive y -axis, and take the cross section containing the x -axis. The top of this cross section is a piece of the parabola $y = 6 - x^2$. (The whole filled ice cream cone is gotten by rotating this cross section about the y -axis.)

What is the volume of the ice cream? (Suggestion: use cylindrical shells.)

8. Rectangles are inscribed as shown in the quarter-circle of radius a , with the point x being chosen randomly on the interval $[0, a]$. Find the average value of their area.

9. Find the approximate value given for the integral below by the trapezoidal rule and also by Simpson’s rule, taking $n = 2$ (i.e., dividing the interval of integration into two equal subintervals):

$$\int_0^{\pi/2} \sin^6 x \, dx$$

Other possible problems: Volumes by vertical slicing 4B, Work problems (P.Set 5)

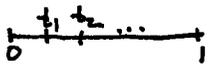
EXAM 3 PRACTICE: SOLUTIONS: FALL 2006

1 a) $\int_0^1 \frac{x dx}{\sqrt{1+3x^2}} = \frac{1}{3} (1+3x^2)^{1/2} \Big|_0^1 = \frac{1}{3} (2-1) = \frac{1}{3}$

b) $\int_{\pi/3}^{\pi/2} \cos^3 x \sin 2x dx = \int_{\pi/3}^{\pi/2} 2 \cos^4 x \sin x dx$
 $= -\frac{2}{5} \cos^5 x \Big|_{\pi/3}^{\pi/2} = -\frac{2}{5} [0 - (\frac{1}{2})^5] = \frac{1}{5 \cdot 16}$

2  Upper sum is sum of circums. rectangles
 $= \sum_{i=1}^n (\frac{i}{n}) \frac{1}{n}$
 $= \frac{1}{n^2} \sum_{i=1}^n i = \frac{n^2+n}{2n^2} = \frac{1}{2} + \frac{1}{2n}$

$\lim_{n \rightarrow \infty} \int_0^1 x dx = \frac{1}{2}$

3  Divide interval into n equal subintervals, length Δt.

* kΔt deposited in the tth time interval grows by year-end to kΔt e^{r(1-t)}
 Total at year-end $\approx \sum_{t=1}^n k e^{r(1-t)} \Delta t$
 As n → ∞, this → $\int_0^1 k e^{r(1-t)} dt$
 [or can make argument using dt instead of Δt: replace Δt by dt in (*), then pass directly to the last line.]

4 a) on [0, 1], sin t < 1
 So $\int_0^2 \sqrt{3+\sin t} dt < \int_0^1 \sqrt{4} dt = 2$.

b) $F'(x) = \sqrt{3+\sin x}$ (2nd F.T.)
 $F''(x) = \frac{\cos x}{2\sqrt{3+\sin x}} > 0$ on [0, 1]

∴ convex (concave up).

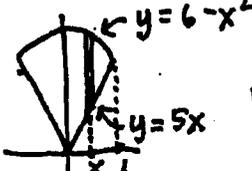
c) Set u = 2t, so du = 2 dt
 $\int_1^2 \sqrt{3+\sin 2t} dt = \int_2^4 \sqrt{3+\sin u} \cdot \frac{1}{2} du$
 $= \frac{1}{2} F(u) \Big|_2^4 = \frac{1}{2} [F(4) - F(2)]$

5 $\int_0^0 f(t) dt = 0 = e^{2 \cdot 0} \cos 0 + C$
 $\therefore C = -1$

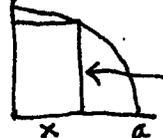
By 2nd F.T.:

$f(x) = D(e^{2x} \cos x - 1)$
 $= 2e^{2x} \cos x - e^{2x} \sin x$

6  horizontal slice →
 $\int_0^{\sqrt{a}} (\pi a^2 - \pi x^2) dy = \pi \int_0^{\sqrt{a}} (a^2 - y^2) dy$
 $= \pi (a^2 y - \frac{y^3}{3}) \Big|_0^{\sqrt{a}} = \pi [a^2 \sqrt{a} - \frac{1}{3} a^{3/2}]$
 $= \pi \cdot \frac{2}{3} a^{3/2}$

7  The strip rotated has volume $2\pi x(6-x^2-5x) dx$

Volume = $2\pi \int_0^1 x(6-x^2-5x) dx$
 $= 2\pi [3x^2 - \frac{1}{4}x^4 - \frac{5}{3}x^3] \Big|_0^1$
 $= 2\pi [3 - \frac{1}{4} - \frac{5}{3}] = 2\pi \cdot \frac{12}{12}$

8  curve: $y = \sqrt{a^2-x^2}$
 $\text{Area} = \int_0^a \sqrt{a^2-x^2} dx$

Average area = $\frac{1}{a} \int_0^a x \sqrt{a^2-x^2} dx$
 $= \frac{1}{a} (a^2-x^2)^{3/2} (-\frac{1}{3}) \Big|_0^a = \frac{1}{3} (a^2)^{3/2} = \frac{a^2}{3}$

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x	0	π/4	π/2
sin x	0	1/√2	1
sin ⁶ x	0	1/8	1

 $\Delta x = \pi/4$
 SIMPSON
 $\int_0^{\pi/2} \sin^6 x dx \approx \frac{\pi/4}{3} (0 + 4 \cdot \frac{1}{8} + 1)$
 TRAPEZOIDAL
 $\approx \frac{\pi}{4} (\frac{1}{2} \cdot 0 + \frac{1}{8} + \frac{1}{2} \cdot 1)$
 $= \frac{\pi}{4} \cdot \frac{5}{8} = \frac{5\pi}{32}$
 $= \frac{\pi}{12} (\frac{3}{2}) = \frac{\pi}{8}$