

Problem 9. (10) Using the trapezoidal rule with three subdivisions ($n = 3$), estimate $\int_0^{\pi/2} \sin^2 x \, dx$. Do the work systematically, making a table of values first.

Problem 10. (15: 7,8) Let $F(x) = \int_0^x e^{-t^2} dt$.

a) Find $F'(1)$ and $F''(1)$.

b) Express $\int_1^2 e^{-u^2/4} du$ in terms of values of $F(x)$.

Problem 11. (15: 7,8) Between the two towers of a suspension bridge, each of the two main cables has the shape of the parabola $y = \frac{1}{10}x^2$ (units are kilometers). The two towers are 2 km. apart; the vertical cables from the main cable to the horizontal roadway are closely and equally spaced.

a) Set up a definite integral which gives the length of each main cable between the two towers.

b) What is the average length (to the nearest meter) of the vertical cables?

Problem 12. (20: 10,10) Evaluate

a) $\int_0^1 \frac{dx}{x^2 + 3x + 2}$; (begin by factoring the denominator).

b) $\int x^2 \ln x \, dx$

Problem 13. (10) Evaluate $\int_0^1 \frac{dx}{(x^2 + 1)^2}$ by making the substitution $x = \tan u$; remember the limits.

Problem 14. (15) Starting at the point where $r = 1$, the point P moves counterclockwise along the polar curve $r = e^{\theta/2\pi}$, in such a way that the line segment OP makes one complete revolution. (Here O denotes the origin.)

Sketch the curve, and find the total area swept out by OP as it makes the revolution.

Problem 15. (15) Evaluate (showing work):

a) $\lim_{x \rightarrow 0} \frac{\sin^2 x}{1 - \cos x}$

b) $\lim_{x \rightarrow 1} \frac{(\ln x)^2}{x - 1}$

c) $\lim_{x \rightarrow \infty} x^2 e^{-x}$

Problem 16. (10) Evaluate $\int_1^{\infty} \frac{dx}{x^{3/2}}$

Problem 17. (10) For what values of p does $\sum_1^{\infty} \frac{n}{\sqrt{4 + n^p}}$ converge? (Indicate reasoning.)

Problem 18. (15: 10,5)

a) Find by differentiating the function $f(x) = \sqrt{1+x}$ the first four non-zero terms of its Taylor series around $x = 0$. (Show work.)

b) Use the correct answer to (a) (or your own answer, if you don't know the correct answer) to calculate $\sqrt{1.2}$ to four decimal places.

Problem 19. (10) Find the Taylor series for $\tan^{-1} x$ around $x = 0$ by using term-by-term differentiation or integration on the appropriate geometric series. Give enough terms to make the pattern clear.

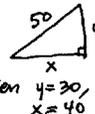
Practice Final SOLUTIONS 18.01

11 a) $\frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{x^4} = \frac{1-2\ln x}{x^3}$
 b) $\frac{1}{2}(3\sin^2 u + 2)^{3/2} \cdot 6\sin u \cos u$
 $= \frac{3\sin u \cos u}{\sqrt{3\sin^2 u + 2}}$
 c) $D^n e^{kx} = k^n e^x$; at 0: k^n

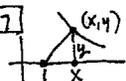
12 $D(x^2 y^2 + y^3) = 2xy^2 + x^2 \cdot 2yy' + 3y^2 y'$
 $= 0$
 At (1,1): $2 + 2y' + 3y' = 0$; $y' = -\frac{2}{5}$
 Eqn: $(y-1) = -\frac{2}{5}(x-1)$ or
 $y = -\frac{2}{5}x + \frac{7}{5}$

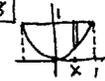
13 $y = \cos^{-1} x$ $y' = \frac{-1}{\sin y}$
 $x = \cos y$ $1 = -\sin y \cdot y'$
 $1 = -\sin y \cdot y' = \frac{-1}{\sqrt{1-\cos^2 y}}$
 $\therefore y' = \frac{-1}{\sqrt{1-x^2}}$
 (use +√ since $\sin y > 0$)

14 $(x^2 + a) \cdot (bx + 2)$ same value at 0 (for continuity)
 $\Rightarrow a = 2$
 same deriv at 0 (for diff'ble)
 $\Rightarrow 2x + 1 = b$ at 0
 $\therefore b = 1$

15  $x^2 + y^2 = 50^2$
 $x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$
 when $y = 30$, $x = 40$
 $40 \cdot \frac{dx}{dt} + 30(-2) = 0$
 $\frac{dx}{dt} = \frac{60}{40} = \frac{3}{2}$ ft/sec

16 $A = 2x \cdot (1-x^2) = 2x - 2x^3$
 $\frac{dA}{dx} = 2 - 6x^2 = 0$ if $x^2 = \frac{1}{3}$
 $x = \frac{1}{\sqrt{3}}$
 Area then is:
 $2 \cdot \frac{1}{\sqrt{3}} (1 - \frac{1}{3}) = \frac{4}{3\sqrt{3}}$

17  $-\frac{1}{y} = \frac{y}{x-1}$
 (negative recip. slope)
 a) $\therefore y' = \frac{1-x}{y}$
 b) $\frac{dy}{dx} = \frac{1-x}{y}$
 $y dy = (1-x) dx$
 $\frac{1}{2} y^2 = x - \frac{1}{2} x^2 + c_1$ [or better: $\frac{1}{2}(1-x)^2 + c_2$]
 $y^2 = 2x - x^2 + c$
 c) $y^2 + (x-1)^2 = c_3$ (completing square)
 circles centered at (1,0)

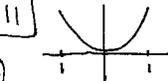
18  Volume = $\int_0^1 2\pi x(1-x^2) dx$
 $= 2\pi (\frac{x^2}{2} - \frac{x^4}{4}) \Big|_0^1$
 $= 2\pi (\frac{1}{4} - \frac{1}{4}) = \frac{\pi}{2}$
 [or: $\pi \cdot 1^2 \cdot 1 - \int_0^1 2\pi x^3 dx$
 vol. cylinder - vol. under curve = $\pi - \frac{\pi}{2} = \frac{\pi}{2}$]

19

| | | | | |
|------------|---|---------|--------------|---------|
| x | 0 | $\pi/6$ | $\pi/3$ | $\pi/2$ |
| $\sin x$ | 0 | $1/2$ | $\sqrt{3}/2$ | 1 |
| $\sin^2 x$ | 0 | $1/4$ | $3/4$ | 1 |


 Trap. rule: $\frac{\pi}{6} (\frac{1}{2} \cdot 0 + \frac{1}{4} + \frac{3}{4} + \frac{1}{2} \cdot 1)$
 $= \frac{\pi}{6} \cdot \frac{3}{2} = \frac{\pi}{4}$

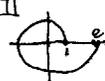
10 $F(x) = \int_0^x e^{-t^2} dt$
 a) $F'(x) = e^{-x^2}$; $F''(x) = -2xe^{-x^2}$
 $F'(1) = \frac{1}{e}$ $F''(1) = -\frac{2}{e}$
 b) $\int_1^2 e^{-u^2/4} du = \int_{1/2}^1 e^{-t^2} \cdot 2 dt$
 Put $t = u/2$
 $dt = \frac{du}{2}$
 $= 2(F(1) - F(1/2))$

11  $y = x^2/10$
 $y' = x/5$
 a) \therefore arc length = $\int_{-1}^1 \sqrt{1 + \frac{x^2}{25}} dx$
 b) average = $\int_0^1 \frac{x^2}{10} dx = \frac{x^3/30} \Big|_0^1 = \frac{1}{30}$
 $= \frac{1}{30} \text{ km} \approx 33 \text{ m}$
 [or = $\frac{1}{2} \int_{-1}^1 x^2 dx = \dots$]

12 $\frac{1}{x^2+3x+2} = \frac{1}{(x+2)(x+1)} = \frac{-1}{x+2} + \frac{1}{x+1}$
 (by cover up)
 a) $\therefore \int (\frac{-1}{x+2} + \frac{1}{x+1}) dx = -\ln|x+2| + \ln|x+1| \Big|_0^1$
 $= -\ln 3 + \ln 2 + \ln 2 - 0$
 $= 2\ln 2 - \ln 3$ (or $\ln \frac{4}{3}$)
 b) $\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$
 int. by parts = $\frac{x^3}{3} \ln x - \frac{x^3}{9} + c$

19 $y = \tan^{-1} x$
 $y' = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \dots$
 Integrate term by term: $\therefore y = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \dots + c$
 $c=0$ since $\tan^{-1} 0 = 0$.

13 $\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4} \frac{\sec^2 u du}{\sec^4 u} = \int_0^{\pi/4} \cos^2 u du$
 $x = \tan u \Rightarrow \int_0^{\pi/4} \frac{1+\cos 2u}{2} du = \frac{u}{2} + \frac{\sin 2u}{4} \Big|_0^{\pi/4}$
 $= \frac{\pi}{8} + \frac{1}{4}$

14  Area = $\frac{1}{2} \int_0^{2\pi} (e^{\theta/2\pi})^2 d\theta$
 $= \frac{1}{2} \int_0^{2\pi} e^{\theta} d\theta = \frac{\pi}{2} e^{\theta/\pi} \Big|_0^{2\pi}$
 $= \frac{\pi}{2} (e^2 - 1)$

15 a) $\lim_{x \rightarrow 0} \frac{\sin^2 x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{2\sin x \cos x}{\sin x} = 2$
 b) $\lim_{x \rightarrow 1} \frac{(\ln x)^2}{x-1} = \lim_{x \rightarrow 1} \frac{2\ln x \cdot \frac{1}{x}}{1} = 0$
 c) $\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$

16 $\int_1^{\infty} \frac{dx}{x^{3/2}} = \int_1^{\infty} x^{-3/2} dx = -2x^{-1/2} \Big|_1^{\infty} = 0 - (-2) = 2$

17 $\frac{n}{\sqrt{4+n^2}} \sim \frac{n}{n^{1/2}} \sim \frac{1}{n^{1/2-1}}$
 $\therefore \sum \frac{n}{\sqrt{4+n^2}}$ converges if $\frac{p}{2} - 1 > 1$
 or $p > 4$

18 $y = (1+x)^{1/2}$ at $x=0$: 1 $\frac{d^2 y}{dx^2}$
 $y' = \frac{1}{2}(1+x)^{-1/2}$ $1/2$ $1/2$
 $y'' = -\frac{1}{2} \cdot \frac{1}{2} (1+x)^{-3/2}$ $-1/4$ $-1/4 \cdot 1/2$
 $y''' = -\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} (1+x)^{-5/2}$ $3/8$ $3/8 \cdot 1/2$
 $\therefore (1+x)^{3/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$
 $(1.2)^{3/2} = 1 + \frac{1}{2} \cdot \frac{3}{10} - \frac{1}{8} \cdot \frac{9}{100} + \frac{1}{16} \cdot \frac{27}{1000}$
 $= 1 + .005 - .0055 + .0005$
 $= 1.0955$