

E. 1801 EXERCISES

7B-3 Prove that if $a_n > 0$ and $\sum_0^\infty a_n$ converges, then $\sum_0^\infty \sin a_n$ also converges.

7B-4 Using the ratio test, or otherwise, determine whether or not each of these series is absolutely convergent. (Note that $0! = 1$.)

- | | | |
|---|--|--|
| a) $\sum_0^\infty \frac{n}{2^n}$ | b) $\sum_0^\infty \frac{2^n}{n!}$ | c) $\sum_1^\infty \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$ |
| d) $\sum_0^\infty \frac{(n!)^2}{(2n)!}$ | e) $\sum_1^\infty \frac{(-1)^n}{\sqrt{n}}$ | f) $\sum_1^\infty \frac{n!}{n^n}$; use $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ |
| g) $\sum_1^\infty \frac{(-1)^n}{n^2}$ | h) $\sum_0^\infty \frac{(-1)^n}{\sqrt{n^2+1}}$ | i) $\sum_0^\infty \frac{n}{n+1}$ |

7B-5 For those series in 7C-4 which are *not* absolutely convergent, tell whether they are conditionally convergent or divergent.

7B-6 By using the ratio test, determine the radius of convergence of each of the following power series.

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|--|---|--|
| a) $\sum_1^\infty \frac{x^n}{n}$ | b) $\sum_1^\infty \frac{2^n x^n}{n^2}$ | c) $\sum_0^\infty n! x^n$ |
| d) $\sum_0^\infty \frac{(-1)^n x^{2n}}{3^n}$ | e) $\sum_0^\infty \frac{(-1)^n x^{2n+1}}{2^n \sqrt{n}}$ | f) $\sum_0^\infty \frac{(2n)! x^{2n}}{(n!)^2}$ |
| g) $\sum_2^\infty \frac{x^n}{\ln n}$ | h) $\sum_0^\infty \frac{2^{2n} x^n}{n!}$ | |

7C: Taylor Approximations and Power Series

7C-1 Using the general formula for the coefficients a_n , find the Taylor series at 0 for the following functions; do the work systematically, calculating in order the $f^{(n)}$, $f^{(n)}(0)$, and then the a_n .

- | | | |
|-------------|---------------|-----------------|
| a) $\cos x$ | b) $\ln(1+x)$ | c) $\sqrt{1+x}$ |
|-------------|---------------|-----------------|

7C-2 Calculate $\sin 1$ using the Taylor series up to the term in x^3 . Estimate the accuracy using the remainder term. (The calculator value is .84147.) Use the remainder term $R_6(x)$, not $R_5(x)$; why?

7C-3 Using the remainder term, tell for what value of n in the approximation

$$e^x \approx 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

the resulting calculation will give e to 3 decimal places (by convention, this means: within .0005).

7C-4 By using the remainder term, tell whether $\cos x \approx 1 - \frac{x^2}{2!}$ will be valid to within .001 over the interval $|x| < .5$.

7C-5 Calculate $\int_0^{.5} e^{-x^2} dx$, using the approximation for e^{-x^2} up to the term in x^4 . Estimate the error, using the correct remainder term (cf. 7B-3), and tell whether the answer will be good to 3 decimal places.

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7B: Convergence Tests

7B-1

a) $\int_0^{\infty} \frac{x}{x^2+4} = \frac{1}{2} \ln(x^2+4) \Big|_0^{\infty} = \infty$; divergent

b) $\int_0^{\infty} \frac{1}{x^2+1} = \tan^{-1} x \Big|_0^{\infty} = \frac{\pi}{2}$; convergent

c) $\int_0^{\infty} \frac{1}{\sqrt{x+1}} = 2(x+1)^{1/2} \Big|_0^{\infty} = \infty$; divergent

d) $\int_1^{\infty} \frac{\ln x}{x} = \frac{1}{2}(\ln x)^2 \Big|_1^{\infty} = \infty$; divergent

e) $\int_2^{\infty} \frac{1}{(\ln x)^p \cdot x} = \frac{(\ln x)^{1-p}}{1-p} \Big|_2^{\infty}$, if $p \neq 1$: divergent if $p < 1$, convergent if $p > 1$

If $p = 1$, $\int_2^{\infty} \frac{dx}{\ln x} = \ln(\ln x) \Big|_2^{\infty} = \infty$. Thus series converges if $p > 1$, diverges if $p \leq 1$.

f) $\int_1^{\infty} \frac{1}{x^p} = \frac{x^{1-p}}{1-p} \Big|_1^{\infty}$, if $p \neq 1$; diverges if $p < 1$, converges if $p > 1$.

If $p = 1$, $\int_1^{\infty} \frac{dx}{x} = \ln x \Big|_1^{\infty} = \infty$; thus series converges if $p > 1$, diverges if $p \leq 1$.

7B-2

a) Convergent; compare with $\sum_1^{\infty} \frac{1}{n^2}$: $\frac{n^2}{n^2+3n} = \frac{1}{1+3/n} \rightarrow 1$ as $n \rightarrow \infty$

b) Divergent; compare with $\sum \frac{1}{n}$: $\frac{n}{n+\sqrt{n}} = \frac{1}{1+1/\sqrt{n}} \rightarrow 1$, as $n \rightarrow \infty$

c) Divergent; compare with $\sum \frac{1}{n}$: $\frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+1/n}} \rightarrow 1$, as $n \rightarrow \infty$

d) Convergent; compare with $\sum_1^{\infty} \frac{1}{n^2}$: $\lim_{n \rightarrow \infty} n^2 \sin\left(\frac{1}{n^2}\right) = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$

e) Convergent; compare with $\sum_1^{\infty} \frac{1}{n^{3/2}}$: $\frac{n^{3/2}\sqrt{n}}{n^2+1} = \frac{n^2}{n^2+1} = \frac{1}{1+1/n^2} \rightarrow 1$ as $n \rightarrow \infty$

f) Divergent, by comparison test: $\frac{\ln n}{n} > \frac{1}{n}$; $\sum_1^{\infty} \frac{1}{n}$ diverges

g) Convergent; compare with $\sum \frac{1}{n^2}$: $\frac{n^2 \cdot n^2}{n^4-1} = \frac{n^4}{n^4-1} \rightarrow 1$ as $n \rightarrow \infty$

h) Divergent; compare with $\sum \frac{1}{4n}$: $\frac{4n \cdot n^3}{4n^4+n^2} = \frac{1}{1+1/4n^2} \rightarrow 1$

7. INFINITE SERIES

7B-3 By the mean-value theorem, $\sin x < x$, if $x > 0$; therefore $\sum_0^{\infty} \sin a_n < \sum_0^{\infty} a_n$; so the series converges by the comparison test.

7B-4

- a) By ratio test, $\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \left(\frac{n+1}{n}\right) \cdot \frac{1}{2} \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$; convergent.
- b) By ratio test, $\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \rightarrow 0$ as $n \rightarrow \infty$; convergent.
- c) By ratio test, $\frac{2^{n+1}}{1 \cdot 3 \cdot \dots \cdot 2n+1} \cdot \frac{1 \cdot 3 \cdot \dots \cdot 2n-1}{2^n} = \frac{2}{2n+1} \rightarrow 0$ as $n \rightarrow \infty$; convergent.
- d) By ratio test, $\frac{(n+1)!^2}{(2n+2)!} \cdot \frac{(2n)!}{n!^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \rightarrow \frac{1}{4}$ as $n \rightarrow \infty$; convergent.
- e) Ratio test fails: $\frac{1}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1} \rightarrow 1$ as $n \rightarrow \infty$; but $\sum \frac{1}{\sqrt{n}}$ diverges; therefore the series is not absolutely convergent.
- f) By ratio test, $\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n} \rightarrow \frac{1}{e} < 1$ as $n \rightarrow \infty$; convergent.
- g) Ratio test fails: $\frac{1}{(n+1)^2} \cdot \frac{n^2}{1} \rightarrow 1$ as $n \rightarrow \infty$; but $\sum \frac{1}{n^2}$ converges; therefore the series is absolutely convergent.
- h) Ratio test fails: $\sum \frac{1}{\sqrt{n^2+1}}$ diverges, by limit comparison with $\sum \frac{1}{n}$; therefore the series is not absolutely convergent.
- i) Ratio test fails: $\sum \frac{n}{n+1}$ diverges by the n -th term test; therefore the series is not absolutely convergent.

7B-5

- e) conditionally convergent: terms alternate in sign, $\frac{1}{\sqrt{n}} \rightarrow 0$, decreasing;
- h) conditionally convergent: terms alternate in sign, $\frac{1}{\sqrt{n^2+1}} \rightarrow 0$, decreasing;
- i) divergent, by the n -th term test: $\lim_{n \rightarrow \infty} \frac{(-1)^n n}{n+1} \neq 0$.

7B-6 In all of these, we are using the ratio test.

- a) $\frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = |x| \cdot \left(\frac{n}{n+1}\right) \rightarrow |x|$ as $n \rightarrow \infty$; converges for $|x| < 1$; $R = 1$
- b) $\frac{2^{n+1}|x|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n|x|^n} = 2|x| \cdot \left(\frac{n}{n+1}\right)^2 \rightarrow 2|x|$ as $n \rightarrow \infty$;
converges for $2|x| < 1$ or $|x| < 1/2$; $R = 1/2$

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c) $\frac{(n+1)!|x|^{n+1}}{n!|x|^n} = (n+1)|x| \rightarrow \infty$ as $n \rightarrow \infty$; converges only for $|x| = 0$; $R = 0$

d) $\frac{|x|^{2(n+1)}}{3^{n+1}} \cdot \frac{3^n}{|x|^{2n}} = \frac{|x|^2}{3} \rightarrow \frac{|x|^2}{3}$ as $n \rightarrow \infty$; converges for $\frac{|x|^2}{3} < 1$,
that is, for $|x| < \sqrt{3}$; $R = \sqrt{3}$

e) $\frac{|x|^{2n+3}}{2^{n+1}\sqrt{n+1}} \cdot \frac{2^n\sqrt{n}}{|x|^{2n+1}} = \frac{|x|^2}{2} \cdot \sqrt{\frac{n}{n+1}} \rightarrow \frac{|x|^2}{2}$ as $n \rightarrow \infty$; converges for $\frac{|x|^2}{2} < 1$ or $|x| < \sqrt{2}$; $R = \sqrt{2}$

f) $\frac{(2n+2)!|x|^{2n+2}}{(n+1)!^2} \cdot \frac{n!^2}{(2n)!|x|^{2n}} = |x|^2 \cdot \frac{(2n+2)(2n+1)}{(n+1)^2} \rightarrow 4|x|^2$ as $n \rightarrow \infty$;
converges for $4|x|^2 < 1$, or $|x| < 1/2$; $R = 1/2$

g) $\frac{|x|^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{|x|^n} = |x| \cdot \frac{\ln n}{\ln(n+1)} \rightarrow |x|$ as $n \rightarrow \infty$; converges for $|x| < 1$; $R = 1$

(By L'Hospital's rule, $\lim_{x \rightarrow \infty} \frac{\ln x}{\ln(x+1)} = \lim_{x \rightarrow \infty} \frac{1/x}{1/(x+1)} = 1$.)

h) $\frac{2^{2n+2}|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{2^{2n}|x|^n} = \frac{2^2|x|}{n+1} \rightarrow 0$ as $n \rightarrow \infty$; converges for all x ; $R = \infty$

~~7C: Taylor Approximations and Series~~

~~7C-1~~

(a) $y = \cos x$ $y' = -\sin x$ $y'' = -\cos x$ $y^{(3)} = \sin x$ $y^{(4)} = \cos x, \dots$
 ~~$y(0) = 1$ $y'(0) = 0$ $y''(0) = -1$ $y^{(3)}(0) = 0$ $y^{(4)}(0) = 1, \dots$~~
 ~~$a_0 = 1$ $a_1 = 0$ $a_2 = -1/2!$ $a_3 = 0$ $a_4 = 1/4!$...~~

~~The pattern then repeats with the higher coefficients, so we get finally~~

~~$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$~~

(b)

~~$y = \ln(1+x)$ $y' = (1+x)^{-1}$ $y'' = -(1+x)^{-2}$ $y^{(3)} = 2!(1+x)^{-3}$ $y^{(4)} = -3!(1+x)^{-4}, \dots$~~
 ~~$y(0) = 0$ $y'(0) = 1$ $y''(0) = -1$ $y^{(3)}(0) = 2!$ $y^{(4)}(0) = -3!, \dots$~~
 ~~$a_0 = 0$ $a_1 = 1$ $a_2 = -1/2$ $a_3 = 1/3$ $a_4 = -1/4 \dots$~~

~~$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1} x^n}{n} + \dots$~~