

~~Monotone Convergence Test. A non-decreasing sequence converges if and only if it is bounded above. In this case, the limit of the sequence is the least upper bound for the sequence. Similarly, a non-increasing sequence converges if and only if it is bounded below and the limit is the greatest lower bound for the sequence.~~

**3. Series.** Given a sequence  $(a_n)_{n \geq 1}$ , there are 2 important related sequences. The first is the *sequence of partial sums*,  $(b_n)_{n \geq 1}$ , defined by,

$$b_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^n a_k.$$

The second is the *sequence of partial absolute sums*,  $(B_n)_{n \geq 1}$ , defined by,

$$B_n = |a_1| + |a_2| + \cdots + |a_n| = \sum_{k=1}^n |a_k|.$$

If the sequence of partial sums  $(b_n)_{n \geq 1}$  converges, the limit is called the *series of*  $(a_n)_{n \geq 1}$ , and is denoted by,

$$\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

In this case it is said *the series*  $\sum_k a_k$  *converges*. If the sequence of partial absolute sums  $(B_n)_{n \geq 1}$  converges, it is said *the series*  $\sum_k a_k$  *converges absolutely*. Although it is not obvious, if the series converges absolutely, then the series converges (this is a basic theorem from course 18.100). If a series converges but does not converge absolutely, sometimes it is said the series *converges conditionally*.

**Examples.** The *harmonic sequence* is the sequence  $a_n = 1/n$ . As will be shown soon, the *harmonic series*  $\sum_n 1/n$  diverges to  $\infty$ . The *alternating harmonic sequence* is,

$$a_n = \frac{(-1)^n}{n}.$$

The *alternating harmonic series*,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

does converge. This will also be shown soon. Since the sequence of partial absolute sums for the alternating sequence equals the sequence of partial sums for the harmonic sequence, the alternating harmonic series does not converge absolutely. It only converges conditionally.

As counter-intuitive as this might sound, the terms in the alternating harmonic series can be rearranged so that the sum converges to any real number you like! This sounds ridiculous: finite sums are independent of the order in which the summands are added, so how could this fail for

infinite sums? The answer is quite simple. Because the harmonic series  $\sum_n 1/n$  diverges, the same is true for  $\sum_{1/2n}$ . Thus, add it up a very large number of *only* the (positive) even terms in the alternating harmonic series to make the partial sum bigger than, say,  $10^6$ . Now add only the first odd term  $-1/2$ . This has a negligible effect. Now add a large number of the remaining even terms to make the partial sum bigger than  $10^7$ . Now add one more odd term,  $-1/3$ . Continuing in this way, eventually every term in the sequence contributes to one of the partial sums. But because we add positive terms with a much higher frequency than negative terms, the sequence of partial sums is diverging to  $+\infty$ . Similarly, we could negative terms with a very high frequency and make the partial sums diverge to  $-\infty$ . Now it is not so surprising that by adding the terms in a careful order, we can make the partial sums converge to any value we like.

The pathology of the preceding paragraph occurs with any conditionally convergent series. It is a very important fact that every absolutely convergent series has only a single limit, independent of the order in which terms are added. For this reason, absolutely convergent series are much more useful than conditionally convergent series.

~~**4. Test for convergence/divergence of series.** If a series  $\sum_n a_n$  converges, then the sequence  $(a_n)$  converges to 0. To see this, denote by  $L$  the limit of the sequence of partial sums  $(b_n)$ . For every positive real number  $\epsilon$ , using  $\epsilon/2$  in the definition of convergence of  $(b_n)$ , there exists an integer  $N$  such that for every  $n \geq N$ ,  $|b_n - L| < \epsilon/2$ . But then for  $n \geq N + 1$ ,~~

~~$$|a_n| = |b_n - b_{n-1}| = |(b_n - L) - (b_{n-1} - L)| \leq |b_n - L| + |b_{n-1} - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$~~

~~Thus the sequence  $(a_n)$  converges to 0. Contrapositively, if the sequence  $(a_n)$  does not converge to 0, then the series  $\sum_n a_n$  diverges. This is the most basic test for divergence of a series. For example, it immediately follows that the series  $\sum_{n=1}^{\infty} (-1)^n$  diverges (arguing the opposite is a favorite pasttime of "mathematical cranks").~~

~~The most basic test for absolute convergence of a sequence follows from the monotone convergence test. The sequence of partial absolute sums,~~

~~$$B_n = \sum_{k=1}^n |a_k|,$$~~

~~is a non-decreasing sequence. Therefore, by the monotone convergence theorem, it converges if and only if it is bounded above. The most common technique for proving the sequence of partial absolute sums is bounded above is by comparing it to a larger series that is known to converge. This gives the following.~~

~~**Comparison Test.** Let  $(a_n)_{n \geq 1}$  and  $(b_n)_{n \geq 1}$  be sequences such that for every index  $n$ ,  $|a_n| \leq |b_n|$ . If the series  $\sum_{n=1}^{\infty} b_n$  converges absolutely, then the series  $\sum_{n=1}^{\infty} a_n$  converges absolutely.~~

~~A number of common convergence tests in calculus textbooks come to nothing more than combining the comparison test with an analysis of the geometric series. Let  $r$  be a real number and let  $(a_n)_{n \geq 0}$  be the geometric sequence,~~

~~$$a_n = r^n, n \geq 0,$$~~