

Lecture 20. November 1, 2005

Practice Problems. Course Reader: 4C-2, 4C-6, 4D-1, 4D-4, 4D-8.

1. Average values. Given a function $f(x)$ defined on some interval $[a, b]$, what is the average value of $f(x)$? A reasonable first approximation is to choose a finite collection of points from $[a, b]$ and compute the average value over those points. Break $[a, b]$ into a union of n subintervals of length $\Delta x = (b - a)/n$. From each interval, choose a point; say x_k^* in the k^{th} interval. For the finitely many values $y_k^* = f(x_k^*)$, the average value is,

$$\text{Average} \approx \frac{1}{n} \sum_{k=1}^n y_k^*.$$

Multiplying and dividing by Δx gives,

$$\text{Average} \approx \frac{1}{n\Delta x} \sum_{k=1}^n y_k^* \Delta x.$$

Now $n\Delta x$ equals $n(a-b)/n$, which is $a-b$. So the average value is,

$$\text{Average} \approx \frac{1}{b-a} \sum_{k=1}^n y_k^* \Delta x.$$

The sum is a Riemann sum. To get better approximations to the true average, increase the number of points n at which $f(x)$ is “sampled”. In the limit, this gives the true average,

$$\text{Average} = \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{k=1}^n y_k^* \Delta x = \int_a^b f(x) dx / (b-a).$$

Example. Under ideal conditions, a wire-producing machine produces wire of uniform radius r_0 . Because of small vibrations in the machine, the actual radius of the wire varies as a function of the length,

$$r(x) = r_0 + A \cos(\omega x).$$

The quantity A is much smaller than r_0 . What is the average radius of the wire?

Because the variation is periodic, the average value over any number of periods equals the average value of one period. In other words, compute the average for the interval $0 \leq x \leq 2\pi/\omega$. The length of this interval is $2\pi/\omega$. Thus the average value is,

$$\text{Average} = \frac{1}{(2\pi/\omega)} \int_0^{2\pi/\omega} r_0 + A \cos(\omega x) dx.$$

Using the Fundamental Theorem of Calculus, this equals,

$$\frac{1}{(2\pi/\omega)} (r_0 x + (A/\omega) \sin(\omega x)) \Big|_0^{2\pi/\omega}.$$

This evaluates to,

$$\frac{1}{(2\pi/\omega)} r_0 (2\pi/\omega) = r_0.$$

Thus, although the radius varies and does not usually equal its ideal value r_0 , the average value is indeed,

$$\text{Average} = r_0.$$

2. Average values: non-uniform distribution. It often happens that the average value of $f(x)$ is desired in a situation where the values $f(x)$ are not all uniformly likely. Typically, the probability that x has value in the range from x_0 to $x_0 + \Delta x$ is approximately,

$$\text{Prob}(x_0 \leq x \leq x_0 + \Delta x) \approx \rho(x_0) \Delta x,$$

for some nonnegative continuous function $\rho(x)$. The function $\rho(x)$ is called a *probability distribution*. Assuming this approximation becomes arbitrarily good as the length Δx approaches zero, the exact probability that x has value in the range x_0 to x_1 is,

$$\text{Prob}(x_0 \leq x \leq x_1) = \int_{x_0}^{x_1} \rho(x) dx.$$

In particular, because x must take value somewhere in the interval $[a, b]$, the total probability is 1. In other words,

$$\int_a^b \rho(x) dx = 1.$$

This is called the *normalization condition*.

The average value is computed as before. But this time, each value $y_k^* = f(x_k^*)$ is weighted by the approximate probability that x takes value in the k^{th} interval, $\rho(x_k^*)\Delta x$. This gives,

$$\text{Average} \approx \sum_{k=1}^n f(x_k^*) \rho(x_k^*) \Delta x.$$

In the limit as n goes to ∞ , this gives the exact average,

$$\text{Average} = \int_a^b f(x) \rho(x) dx.$$

It must be noted, the probability distribution $\rho(x)$ often does not satisfy the normalization condition. In this case, the formula above is wrong. But it is easily corrected,

$$\text{Average} = \left(\int_a^b f(x) \rho(x) dx \right) / \left(\int_a^b \rho(x) dx \right).$$

Example. A particle is fired through a slit and strikes a screen on the other side. Measuring the position on the screen so that the origin is the closest point on the screen to the slit, the probability distribution is empirically observed to be,

$$\rho(x) = C e^{-x^2/2\sigma^2},$$

where σ is a constant determining the “width” of the probability distribution, and C is an undetermined normalization constant. What is the average distance of the particle from the center of the screen? Assume the particle lies in an interval $[-R, R]$, where R is very large.

Remark. This differs from the formula given in lecture, which was $C e^{-x^2/2\sigma}$ for a particular choice of σ . The formula given here is more “standard”. I apologize for any confusion.

The distance function is,

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

According to the formula, the average value is,

$$\left(\int_{-R}^R f(x)\rho(x)dx\right)/\left(\int_{-R}^R \rho(x)dx\right).$$

The numerator is,

$$\int_{-R}^R |x|Ce^{-x^2/2\sigma^2} dx.$$

It is easiest to compute this by breaking it into a sum of 2 integrals,

$$\int_{-R}^0 (-x)Ce^{-x^2/2\sigma^2} dx + \int_0^R (+x)Ce^{-x^2/2\sigma^2} dx.$$

Make the substitution $u = -x^2/2\sigma^2$, $du = (-x/\sigma^2)dx$ to reduce this to,

$$\int_{-R^2/2\sigma^2}^0 Ce^u(\sigma^2 du) + \int_0^{-R^2/2\sigma^2} Ce^u(-\sigma^2 du) = 2 \int_{-R^2/2\sigma^2}^0 C\sigma^2 e^u du.$$

Using the Fundamental Theorem of Calculus, this equals,

$$2C\sigma^2 (e^u|_{-R^2/2\sigma^2}^0 = 2C\sigma^2(1 - e^{-R^2/2\sigma^2}).$$

As R becomes large, the quantity $e^{-R^2/2\sigma^2}$ becomes vanishingly small. Thus, in the limit as R tends to ∞ , the numerator equals,

$$\lim_{R \rightarrow \infty} \int_{-R}^R |x|Ce^{-x^2/2\sigma^2} dx = 2C\sigma^2.$$

Unfortunately, this is not an answer, because the normalization constant C is unknown. The normalization condition is that,

$$C \lim_{R \rightarrow \infty} \int_{-R}^R e^{-x^2/2\sigma^2} dx = 1.$$

Simplify this by making the substitution, $u = x/\sigma$, $du = dx/\sigma$, and $Q = R/\sigma$ to get,

$$C \lim_{R \rightarrow \infty} \int_{-R/\sigma}^{R/\sigma} e^{-u^2/2} \sigma du = C\sigma \lim_{Q \rightarrow \infty} \int_{-Q}^Q e^{-u^2/2} du.$$

Notice the limit,

$$\lim_{Q \rightarrow \infty} \int_{-Q}^Q e^{-u^2/2} du,$$

does not depend on σ . It is simply some number. Denoting this number by $1/C_1$, the normalization condition is,

$$C\sigma/C_1 = 1.$$

The solution is that $C = C_1/\sigma$. Plugging this into the formula above, the average distance is,

$$\text{Average distance} = 2C_1\sigma,$$

where,

$$1/C_1 = \lim_{Q \rightarrow \infty} \int_{-Q}^Q e^{-u^2/2} du.$$

There is a beautiful argument that,

$$C_1 = 1/\sqrt{2\pi}.$$

Unfortunately, we cannot yet prove this. Taking it as true gives the final answer,

$$\text{Average distance} = 2\sigma/\sqrt{2\pi}.$$

3. Volumes of solids of revolution: the shell method. An alternative to the disk and washer method is the shell method. A *shell* is the region between 2 cylinders of the same height. If the average radius of the cylinders is r , if the width of the region is w and if the height of the cylinders is h , then the approximate volume of the shell is,

$$\text{Volume} \approx \text{Circumference} \times \text{height} \times \text{width} = 2\pi r h w.$$

Take the plane region bounded by $x = a$, $x = b$, the x -axis and the curve $y = f(x)$. Revolve this region about the y -axis. (**Please note:** In the disk and washer method, the region was revolved about the x -axis.) To compute the corresponding volume, approximate the region obtained from x to $x + dx$ as a shell. The radius of the shell is x . The height of the shell is $y = f(x)$. The width of the shell is dx . Therefore the differential element of volume is,

$$dV = (2\pi x)(f(x))dx.$$

Integrating gives the volume,

$$V = \int_{x=a}^{x=b} 2\pi x f(x) dx.$$

Example. The dog dish revisited. The main part of a dog dish is a solid of revolution whose radial cross-section is a triangle of height H whose base has inner radius R_i and outer radius R_o . Find the volume of material used to make the dog dish.

The volume was computed using the washer method. This time it will be computed using the shell method. The triangular region is the union of two regions. The first region is bounded by $x = R_i$, $x = (R_i + R_o)/2$, the x -axis, and the line segment,

$$y = \frac{2H}{R_o - R_i}(x - R_i).$$

The second region is bounded by $x = (R_i + R_o)/2$, $x = R_o$, the x -axis, and the line segment,

$$y = \frac{2H}{R_o - R_i}(R_o - x).$$

By the shell method, the volume of the solid of revolution obtained from the first region is,

$$V_1 = \int_{x=R_i}^{x=(R_i+R_o)/2} (2\pi x) \frac{2H}{R_o - R_i} (x - R_i) dx = \frac{4\pi H}{R_o - R_i} \int_{x=R_i}^{x=(R_i+R_o)/2} x^2 - R_i x dx.$$

This becomes simpler to deal with after the substitution $u = -x + (R_i + R_o)/2$, $du = -dx$. The new integral is,

$$\begin{aligned} V_1 &= \frac{4\pi H}{R_o - R_i} \int_{u=(R_o-R_i)/2}^{u=0} (-u + (R_o + R_i)/2)(-u + (R_o - R_i)/2)(-du) \\ &= \frac{4\pi H}{R_o - R_i} \int_{u=0}^{u=(R_o-R_i)/2} (-u + (R_o + R_i)/2)(-u + (R_o - R_i)/2) du. \end{aligned}$$

By the shell method, the volume of the solid of revolution obtained from the second region is,

$$V_2 = \int_{x=(R_o+R_i)/2}^{x=R_o} (2\pi x) \frac{2H}{R_o - R_i} (R_o - x) dx = \frac{4\pi H}{R_o - R_i} \int_{x=(R_o+R_i)/2}^{x=R_o} x(R_o - x) dx.$$

Believe it or not, this will be simpler to deal with after the substitution $u = x - (R_o + R_i)/2$, $du = dx$. The new integral is

$$V_2 = \frac{4\pi H}{R_o - R_i} \int_{u=0}^{u=(R_o-R_i)/2} (u + (R_o + R_i)/2)(-u + (R_o - R_i)/2) du.$$

Notice how similar are the integrals for V_1 and V_2 . They have the same fraction in front of the integral, and they have the same limits of integration. Thus, the sum of the 2 volumes is,

$$V = V_1 + V_2 =$$

$$\frac{4\pi H}{R_o - R_i} \int_{u=0}^{u=(R_o-R_i)/2} [(-u + (R_o + R_i)/2)(-u + (R_o - R_i)/2)] + [(u + (R_o + R_i)/2)(-u + (R_o - R_i)/2)] du.$$

Since both terms in the integrand have the factor $(-u + (R_o - R_i)/2)$, this can be factored to give,

$$V = \frac{4\pi H}{R_o - R_i} \int_{u=0}^{u=(R_o-R_i)/2} [(-u + (R_o + R_i)/2) + (u + (R_o + R_i)/2)](-u + (R_o - R_i)/2) du.$$

Of course the term in square brackets is simply $R_o + R_i$. So the total volume is,

$$V = \frac{4\pi H}{R_o - R_i} \int_{u=0}^{u=(R_o-R_i)/2} (R_o + R_i)(-u + (R_o - R_i)/2) du.$$

By the Fundamental Theorem of Calculus, this equals,

$$\frac{4\pi H}{R_o - R_i}(R_o + R_i) \left(\frac{-u^2}{2} + \frac{(R_o - R_i)u}{2} \right) \Big|_0^{(R_o - R_i)/2}.$$

This evaluates to,

$$\frac{4\pi H}{R_o - R_i}(R_o + R_i) \frac{(R_o - R_i)^2}{8}.$$

This simplifies to give,

$$V = \pi H(R_o - R_i)(R_o + R_i)/2 = \pi(R_o^2 - R_i^2)H/2.$$

This is precisely the same answer as computed using the washer method. Please observe though, how much more effort was required for the shell method. The lesson is, if you have an alternative between the disk method and the shell method, consider carefully which method requires less effort before committing to one or the other.