

**Lecture 13.** October 13, 2005

~~**Homework.** Problem Set 4 Part I: (a) and (b); Part II: Problem 3.~~

~~**Practice Problems.** Course Reader: 3A-1, 3A-2, 3A-3.~~

**1. Differentials.** An alternative notation for derivatives is *differential notation*. The differential notation,

$$dF(x) = f(x)dx,$$

is shorthand for the sentence “The derivative of  $F(x)$  with respect to  $x$  equals  $f(x)$ .” Formally, this is related to the Leibniz notation for the derivative,

$$\frac{dF}{dx}(x) = f(x),$$

which means the same thing as the differential notation. It may look like the first and second equation are obtained by dividing and multiplying by the quantity  $dx$ . It is crucial to remember that  $dF/dx$  is **not a fraction**, although the notation suggests otherwise.

In differential notation, some derivative rules have a very simple form, and are thus easier to remember. Here are a few derivative rules in differential notation.

$$\begin{aligned} dF(x) &= F'(x)dx \\ d(F(x) + G(x)) &= dF(x) + dG(x) \\ d(cF(x)) &= cdF(x) \\ d(F(x)G(x)) &= G(x)dF(x) + F(x)dG(x) \\ d(F(x)/G(x)) &= 1/(G(x))^2(G(x)dF(x) - F(x)dG(x)) \end{aligned}$$

The chain rule has a particularly simple form,

$$d(F(u)) = \frac{dF}{du} du = \frac{dF}{du} \frac{du}{dx} dx.$$

**Example.** Using differential notation, the derivative of  $\sin(\sqrt{x^2 + 1})$  is,

$$\begin{aligned} d \sin((x^2 + 1)^{1/2}) &= \cos((x^2 + 1)^{1/2})d(x^2 + 1)^{1/2} = \cos((x^2 + 1)^{1/2})\left(\frac{1}{2}(x^2 + 1)^{-1/2}\right)d(x^2 + 1) = \\ &= \cos((x^2 + 1)^{1/2})\frac{1}{2}(x^2 + 1)^{-1/2}(2xdx) = x(x^2 + 1)^{-1/2} \cos((x^2 + 1)^{1/2})dx. \end{aligned}$$

~~2. Antidifferentiation.~~ Recall, the basic problem of differentiation is the following.

~~Problem (Differentiation).~~ Given a function  $F(x)$ , find the function  $f(x)$  satisfying  $\frac{dF}{dx} = f(x)$ .

~~The basic problem of antidifferentiation is the inverse problem.~~

~~Problem (Antidifferentiation).~~ Given a function  $f(x)$ , find a function  $F(x)$  satisfying  $\frac{dF}{dx} = f(x)$ .

~~A function  $F(x)$  solving the problem is called an antiderivative of  $f(x)$ , or sometimes an indefinite integral of  $f(x)$ . The notation for this is,~~

$$F(x) = \int f(x)dx.$$

~~The expression  $f(x)$  is called the integrand. It is important to note, if  $F(x)$  is one antiderivative of  $f(x)$ , then for each constant  $C$ ,  $F(x) + C$  is also an antiderivative of  $f(x)$ . The constant  $C$  is called a constant of integration.~~

~~In a sense that can be made precise, the problem of differentiation has a complete solution whenever  $F(x)$  is a “simple expression”, i.e., a function built from the differentiable functions we have seen so far. Unfortunately, for very many simple functions  $f(x)$ , no antiderivative of  $f(x)$  has a simple expression. In large part, this is what makes antidifferentiation difficult. Luckily, many of the most important simple functions  $f(x)$  do have an antiderivative with a simple expression. One goal of this unit is to learn how to recognize when a simple antiderivative exists, and some tools to compute the antiderivative.~~

~~3. Antidifferentiation. Guess-and-check.~~ The main technique for antidifferentiation is educated guessing.

~~Example.~~ Find an antiderivative of  $f(x) = x^2 + 2x + 1$ . Since the derivative of  $x^n$  is  $nx^{n-1}$ , it is reasonable to guess there is an antiderivative of the form  $F(x) = Ax^3 + Bx^2 + Cx$ . Differentiation gives,

$$\frac{dF}{dx} = 3Ax^2 + 2Bx + C.$$

Thus,  $F(x)$  is an antiderivative of  $f(x)$  if and only if,

$$3A = 1, \quad 2B = 2, \quad \text{and } C = 1.$$

This gives an antiderivative,

$$\int (x^2 + 2x + 1)dx = \frac{1}{3}x^3 + x^2 + x + E,$$