Problem 2(5 points)

(a)(2 points) What does the chain rule say if $y = x^a$ and $u = y^b$? The constants a and b are fractions.

(b)(3 points) Using the chain rule, give a very short explanation of the formula from Problem 3, Part II of Problem Set 1.

Problem 3(10 points) A bank offers *savings accounts* and *loans*. For an initial deposit of A dollars in a savings account with continuously compounded interest at an annual rate a, after t years the bank owes the customer $A(1+a)^t$ dollars (neglecting fees). For an initial loan of B dollars with continuously compounded interest at an annual rate b, after t years the customer owes the bank $B(1+b)^t$ dollars (neglecting fees). To make a profit, the bank sets rate b, the interest rate for loans, higher than rate a, the interest rate for savings. To simplify computations, introduce $\alpha = \ln(1+a)$ and $\beta = \ln(1+b)$.

Customer 1 deposits A dollars in a savings account. The bank immediately loans a smaller amount of B dollars to Customer 2. After t years, the bank's net gain from the two transactions together is,

$$G(t) = Be^{\beta t} - Ae^{\alpha t}. \tag{1}$$

In the long run, which is to say, when t is very large, $G(t)$ is positive and the bank has made a gain. However, for t small, $G(t)$ is negative and the bank has a net liability,

$$L(t) = -G(t) = Ae^{\alpha t} - Be^{\beta t}. \tag{2}$$

The liability for the savings account alone is,

$$M(t) = Ae^{\alpha t}. \tag{3}$$

In these equations, A, B, α and β are positive constants, and t is the independent variable.

(a)(5 points) Find the moment $t = T$ when the derivative $L'(T)$ equals 0. Assume that αA is greater than βB. Also, leave your answer in the form,

$$e^{(\beta-\alpha)T} = \text{something}.$$

Remark. After Lecture 10, we will learn that T is the moment when $L(t)$ has its largest value. In other words, if at time t Customer 1 withdraws all money, and Customer 2 repays all money, the bank loses the maximum amount when $t = T$.

(b)(5 points) Consider the ratio $L(t)/M(t)$. Using your answer to **(a)**, determine $L(T)/M(T)$. Simplify your answer as much as possible. How does this ratio depend on the amounts A and B?

Problem 4(10 points) Let A, β, ω and t_0 be positive constants. Let $f(t)$ be the function,

$$f(t) = Ae^{-\beta t}\cos(\omega(t - t_0)).$$

(a)(5 points) Compute $f'(t)$ and $f''(t)$. Simplify your answer as much as possible.

(b)(5 points) Using your answer to **(a)**, find nonzero constants c_0, c_1 and c_2 for which the function

$$c_2 f''(t) + c_1 f'(t) + c_0 f(t),$$

always equals 0.

It is worth remarking that to make T small, the bank may maximize the fraction B/A of money loaned to money deposited and the bank may maximize the fraction β/α (although if β is too high or α too low, customers are discouraged from using the bank).

(b)(5 points) Consider the ratio $L(t)/M(t)$. Using your answer to **(a)**, determine $L(T)/M(T)$. Simplify your answer as much as possible. How does this ratio depend on the amounts A and B?

Solution to (b) The ratio $L(t)/M(T)$ equals,

$$\frac{L(t)}{M(t)} = \frac{Ae^{\alpha t} - Be^{\beta t}}{Ae^{\alpha t}} = 1 - \frac{B}{A}e^{(\beta-\alpha)t}.$$

By the formula from the **Solution to (a)**, $e^{(\beta-\alpha)T}$ equals $(\alpha A)/(\beta B)$. Plugging this in,

$$\frac{L(T)}{M(T)} = 1 - \frac{B}{A}\frac{\alpha A}{\beta B} = 1 - (\alpha/\beta).$$

In particular, this is independent of A and B.

Remark. From the formulas, the bank's strategy is clear. First, adjust the ratio β/α to the highest level allowed by law and compatible with the market's demands. Then, for fixed α and β, the ratio $L(T)/M(T)$ is independent of A and B. So the maximal liability $L(T)$ is proportional to $M(T)$. Since $M(T)$ is an increasing function, the strategy is to minimize T, by maximizing the ratio B/A. (Of course this ratio will always be less than 1, since some fraction of all capital goes to the federal reserve, some fraction is used to cover operating expenses, etc.)

Problem 4(10 points) Let A, β, ω and t_0 be positive constants. Let $f(t)$ be the function,

$$f(t) = Ae^{-\beta t}\cos(\omega(t - t_0)).$$

(a)(5 points) Compute $f'(t)$ and $f''(t)$. Simplify your answer as much as possible.

Solution to (a) Let s equal $t - t_0$. Then $f(t) = g(s)$, where,

$$g(s) = Be^{-\beta s}\cos(\omega s),$$

and $B = Ae^{-\beta t_0}$. The derivative ds/dt equals 1. Thus, according to the chain rule,

$$\frac{df}{dt} = \frac{dg}{ds}.$$

Using the product rule and the chain rule, this equals,

$$\begin{gathered}\tfrac{dg}{ds} = B(e^{-\beta s})'\cos(\omega s) + Be^{-\beta s}(\cos(\omega s))' = \\ B(-\beta e^{-\beta s})\cos(\omega s) + Be^{-\beta s}(-\omega\sin(\omega s)) = \\ -B\beta e^{-\beta s}\cos(\omega s) - B\omega e^{-\beta s}\sin(\omega s).\end{gathered}$$

In particular, when B equals 1, this gives,

$$\frac{d}{ds}(e^{-\beta s}\cos(\omega s)) = -\beta e^{-\beta s}\cos(\omega s) - \omega e^{-\beta s}\sin(\omega s)).$$

The second derivative $g''(s)$ involves the derivative of $e^{-\beta s}\cos(\omega s)$, but it is also involves the derivative of $e^{-\beta s}\sin(\omega s)$. Using the chain rule and the product rule,

$$\frac{d}{ds}(e^{-\beta s}\sin(\omega s)) = (e^{-\beta s})'\sin(\omega s) + e^{-\beta s}(\sin(\omega s))' = \\ -\beta e^{-\beta s}\sin(\omega s) + \omega e^{-\beta s}\cos(\omega s).$$

As above,

$$\frac{d^2 f}{dt^2} = \frac{d^2 g}{ds^2}.$$

This is,

$$\frac{d}{ds}(\frac{dg}{ds}) = -B\beta[e^{-\beta s}\cos(\omega s)]' - B\omega[e^{-\beta s}\sin(\omega s)]' = \\ -B\beta[-\beta e^{-\beta s}\cos(\omega s) - \omega e^{-\beta s}\sin(\omega s)] - B\omega[-\beta e^{-\beta s}\sin(\omega s) + \omega e^{-\beta s}\cos(\omega s)] = \\ (B\beta^2 - B\omega^2)e^{-\beta s}\cos(\omega s) + (B\beta\omega + B\beta\omega)e^{-\beta s}\sin(\omega s) = \\ B(\beta^2 - \omega^2)e^{-\beta s}\cos(\omega s) + 2B\beta\omega e^{-\beta s}\sin(\omega s).$$

Back-substituting $s = t - t_0$ and $Be^{-\beta s} = Ae^{-\beta t}$ gives,

$$f'(t) = \boxed{-A\beta e^{-\beta t}\cos(\omega(t-t_0)) - A'\omega e^{-\beta t}\sin(\omega(t-t_0)),}$$

and,

$$f''(t) = \boxed{A(\beta^2 - \omega^2)e^{-\beta t}\cos(\omega(t-t_0)) + 2A\beta\omega e^{-\beta t}\sin(\omega(t-t_0)).}$$

(b)(5 points) Using your answer to **(a)**, find nonzero constants c_0, c_1 and c_2 for which the function

$$c_2 f''(t) + c_1 f'(t) + c_0 f(t),$$

always equals 0.

Solution to (b) From the **Solution to (a)**, $f'(t)+\beta f(t)$ equals $-Ae^{-\beta t}(\omega\sin(\omega(t-t_0)))$. Plugging this into the formula for $f''(t)$ gives,

$$f''(t) = (\beta^2 - \omega^2)f(t) - 2\beta(f'(t) + \beta(f)) = -2\beta f' - (\beta^2 + \omega^2)f(t).$$

Simplifying gives,

$$f''(t) + 2\beta f'(t) + (\omega^2 + \beta^2)f(t) = 0.$$

In fact, every solution is of the form,

$$\boxed{c_2 = c, c_1 = 2\beta c, c_0 = \omega^2 + \beta^2,}$$

for some nonzero c.