

11. Exponential and Logarithms: Calculus

11-1 Calculate the derivatives

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|--------------------------|-------------------|-----------------------|
| a) xe^x | b) $(2x-1)e^{2x}$ | c) e^{-x^2} |
| d) $x \ln x - x$ | e) $\ln(x^2)$ | f) $(\ln x)^2$ |
| g) $(e^{x^2})^2$ | h) x^x | i) $(e^x + e^{-x})/2$ |
| j) $(e^x - e^{-x})/2$ | k) $\ln(1/x)$ | l) $1/\ln x$ |
| m) $(1 - e^x)/(1 + e^x)$ | | |

11-2 Graph the function $y = (e^x + e^{-x})/2$.11-3 a) Evaluate $\lim_{n \rightarrow \infty} n \ln(1 + \frac{1}{n})$. Hint: Let $h = 1/n$, and use $(d/dx) \ln(1+x)|_{x=0} = 1$.b) Deduce that $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$.11-4 Using $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$, calculate

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|---|---|--|
| a) $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^{3n}$ | b) $\lim_{n \rightarrow \infty} (1 + \frac{2}{n})^{5n}$ | c) $\lim_{n \rightarrow \infty} (1 + \frac{1}{2n})^{5n}$ |
|---|---|--|

11-5* If you invest P dollars at the annual interest rate r , then after one year the interest is $I = rP$ dollars, and the total amount is $A = P + I = P(1 + r)$. This is *simple interest*.For *compound interest*, the year is divided into k equal time periods and the interest is calculated and added to the account at the end of each period. So at the end of the first period, $A = P(1 + r(\frac{1}{k}))$; this is the new amount for the second period, at the end of which $A = P(1 + r(\frac{1}{k}))(1 + r(\frac{1}{k}))$, and continuing this way, at the end of the year the amount is

$$A = P \left(1 + \frac{r}{k}\right)^k.$$

The compound interest rate r thus earns the same in a year as the simple interest rate of

$$\left(1 + \frac{r}{k}\right)^k - 1;$$

this equivalent simple interest rate is in bank jargon the "annual percentage rate" or APR.³a) Compute the APR of 5% compounded monthly, daily,⁴ and continuously. Continuous compounding means the limit as k tends to infinity.b) As in part (a), compute the APR of 10% compounded monthly, biweekly ($k=26$), daily, and continuously. (We have thrown in the biweekly rate because loans can be paid off biweekly.)

³Banks are required to reveal this so-called APR when they offer loans. The APR also takes into account certain bank fees known as points. Unfortunately, not all fees are included in it, and the true costs are higher if the loan is paid off early.

⁴For daily compounding assume that the year has 365 days, not 365.25. Banks are quite careful about these subtle differences. If you look at official tables of rates from precalculator days you will find that they are off by small amounts because U.S. regulations permitted banks to pretend that a year has 360 days.

1. DIFFERENTIATION

1H-7 a) If I_1 is the intensity of the jet and I_2 is the intensity of the conversation, then

$$\log_{10}(I_1/I_2) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}(I_1/I_0) - \log_{10}(I_2/I_0) = 13 - 6 = 7$$

Therefore, $I_1/I_2 = 10^7$.

b) $I = C/r^2$ and $I = I_1$ when $r = 50$ implies

$$I_1 = C/50^2 \implies C = I_1 50^2 \implies I = I_1 50^2 / r^2$$

This shows that when $r = 100$, we have $I = I_1 50^2 / 100^2 = I_1 / 4$. It follows that

$$10 \log_{10}(I/I_0) = 10 \log_{10}(I_1/4I_0) = 10 \log_{10}(I_1/I_0) - 10 \log_{10} 4 \approx 130 - 6.0 \approx 124$$

The sound at 100 meters is 124 decibels.

The sound at 1 km has 1/100 the intensity of the sound at 100 meters, because $100\text{m}/1\text{km} = 1/10$.

$$10 \log_{10}(1/100) = 10(-2) = -20$$

so the decibel level is $124 - 20 = 104$.

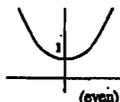
1I. Exponentials and Logarithms: Calculus

1I-1 a) $(x+1)e^x$ b) $4xe^{2x}$ c) $(-2x)e^{-x^2}$ d) $\ln x$ e) $2/x$ f) $2(\ln x)/x$ g) $4xe^{2x^2}$

h) $(x^x)' = (e^{x \ln x})' = (x \ln x)' e^{x \ln x} = (\ln x + 1)e^{x \ln x} = (1 + \ln x)x^x$

i) $(e^x - e^{-x})/2$ j) $(e^x + e^{-x})/2$ k) $-1/x$ l) $-1/x(\ln x)^2$ m) $-2e^x/(1+e^x)^2$

1I-2



1I-3 a) As $n \rightarrow \infty$, $h = 1/n \rightarrow 0$.

$$n \ln\left(1 + \frac{1}{n}\right) = \frac{\ln(1+h)}{h} = \frac{\ln(1+h) - \ln(1)}{h} \xrightarrow{h \rightarrow 0} \frac{d}{dx} \ln(1+x) \Big|_{x=0} = 1$$

Therefore,

$$\lim_{n \rightarrow \infty} n \ln\left(1 + \frac{1}{n}\right) = 1$$

b) Take the logarithm of both sides. We need to show

$$\lim_{n \rightarrow \infty} \ln\left(1 + \frac{1}{n}\right)^n = \ln e = 1$$

But

$$\ln\left(1 + \frac{1}{n}\right)^n = n \ln\left(1 + \frac{1}{n}\right)$$

so the limit is the same as the one in part (a).

S. 18.01 SOLUTIONS TO EXERCISES

1I-4 a)

$$\left(1 + \frac{1}{n}\right)^{3n} = \left(\left(1 + \frac{1}{n}\right)^n\right)^3 \rightarrow e^3 \text{ as } n \rightarrow \infty,$$

b) Put $m = n/2$. Then

$$\left(1 + \frac{2}{n}\right)^{5n} = \left(1 + \frac{1}{m}\right)^{10m} = \left(\left(1 + \frac{1}{m}\right)^m\right)^{10} \rightarrow e^{10} \text{ as } m \rightarrow \infty$$

c) Put $m = 2n$. Then

$$\left(1 + \frac{1}{2n}\right)^{5n} = \left(1 + \frac{1}{m}\right)^{5m/2} = \left(\left(1 + \frac{1}{m}\right)^m\right)^{5/2} \rightarrow e^{5/2} \text{ as } m \rightarrow \infty$$

1J. Trigonometric functions

~~1J-1 a) $10x \cos(5x^2)$ b) $6 \sin(3x) \cos(3x)$ c) $-2 \sin(2x)/\cos(2x) = -2 \tan(2x)$~~

~~d) $-2 \sin x / (2 \cos x) = -\tan x$. (Why did the factor 2 disappear? Because $\ln(2 \cos x) = \ln 2 + \ln(\cos x)$, and the derivative of the constant $\ln 2$ is zero.)~~

~~e) $\frac{x \cos x - \sin x}{x^2}$ f) $(1 + y') \sin(x + y)$ g) $\sin(x + y)$ h) $2 \sin x \cos x \sin^2 x$~~

~~i) $\frac{(x^2 \sin x)'}{x^2 \sin x} = \frac{2x \sin x + x^2 \cos x}{x^2 \sin x} = \frac{2}{x} + \cot x$. Alternatively,~~

$$\ln(x^2 \sin x) = \ln(x^2) + \ln(\sin x) = 2 \ln x + \ln \sin x$$

~~Differentiating gives $\frac{2}{x} + \frac{\cos x}{\sin x} = \frac{2}{x} + \cot x$~~

~~j) $2e^{2x} \sin(10x) + 10e^{2x} \cos(10x)$ k) $6 \tan(3x) \sec^2(3x) = 6 \sin x / \cos^3 x$~~

~~l) $-x(1 - x^2)^{-1/2} \sec(\sqrt{1 - x^2}) \tan(\sqrt{1 - x^2})$~~

~~m) Using the chain rule repeatedly and the trigonometric double angle formulas,~~

$$\langle \cos^2 x - \sin^2 x \rangle' = -2 \cos x \sin x - 2 \sin x \cos x = -4 \cos x \sin x;$$

$$(2 \cos^2 x)' = -4 \cos x \sin x;$$

$$\langle \cos(2x) \rangle' = -2 \sin(2x) = -2(2 \sin x \cos x).$$

~~The three functions have the same derivative, so they differ by constants. And indeed,~~

$$\cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1, \quad (\text{using } \sin^2 x = 1 - \cos^2 x).$$

~~n)~~

$$5(\sec(5x) \tan(5x)) \tan(5x) + 5(\sec(5x)(\sec^2(5x))) = 5 \sec(5x)(\sec^2(5x) + \tan^2(5x))$$

Other forms: $5 \sec(5x)(2 \sec^2(5x) - 1); \quad 10 \sec^3(5x) - 5 \sec(5x)$