

~~Lecture 10. September 30, 2005~~

~~Homework. Problem Set 3 Part I: (a)–(f). Part II: Problems 1, 2 and 3.~~

~~Practice Problems. Course Reader: 2C-5, 2C-10, 2C-12, 2D-3, 2D-4.~~

~~1. Asymptotes. An asymptote describes the behavior of the graph of  $y = f(x)$  as it becomes unbounded, in some sense. There are two main examples. The function  $f$  has a vertical asymptote  $x = a$  if at least 1 of the following holds,~~

$$\lim_{x \rightarrow a^-} f(x) = +\infty, \lim_{x \rightarrow a^-} f(x) = -\infty, \lim_{x \rightarrow a^+} f(x) = +\infty, \lim_{x \rightarrow a^+} f(x) = -\infty.$$

~~In each case, the graph of  $y = f(x)$  becomes unbounded, and becomes arbitrarily close to the line  $x = a$ . If  $x = a$  is a vertical asymptote, then  $f(x)$  has an infinite discontinuity at  $x = a$ .~~

~~The function  $f$  has a horizontal asymptote  $y = b$  if at least 1 of the following holds,~~

$$\lim_{x \rightarrow +\infty} f(x) = b, \lim_{x \rightarrow -\infty} f(x) = b.$$

~~In other words, the graph of  $y = f(x)$  becomes arbitrarily close to the line  $y = b$  as  $x$  approaches either  $+\infty$  or  $-\infty$ .~~

~~Example. For the function  $y = (x^3 + x)/(x^2 - 1) = x(x^2 + 1)/(x^2 - 1)$ , the lines  $x = 1$  and  $x = -1$  are vertical asymptotes. There is no horizontal asymptote. However, the graph of  $y$  is asymptotic to the line  $y = x$ . This was not discussed in lecture. A pair of functions  $f$  and  $g$  are asymptotic to each other if the line  $y = 0$  is a horizontal asymptote of  $f - g$ .~~

**2. Applied maximum/minimum problems.** Using the First Derivative Test, the maximum and minimum of many functions can be computed. This is very important in applications.

**Example.** Two long walls meet at right angles making a corner. Using a length of 10 meters of fence to form the other 2 sides of a rectangle, what is the largest area that can be enclosed in this corner?

**Step 1. Identify parameters.** A *parameter* is a constant or variable. The constant in this problem is 10 meters. Two variables are the length  $l$  of one side of the rectangle, and the width  $w$  of the remaining side of the rectangle.

**Step 2. Draw a diagram.** This was done in lecture.

**Step 3. Find the quantity to be maximized or minimized.** The quantity to be maximized is the area  $A$  of the rectangle. Since the area is the product of the length and width,  $A$  equals  $lw$ .

**Step 4. Use the constraints to eliminate variables.** The constraint is that the total length of fence is 10 meters. Thus  $l + w$  equals 10. This is used to eliminate  $w$ ,

$$w = 10 - l.$$

Making this substitution,  $A$  is now a function of  $l$  alone,

$$A(l) = lw(l) = l(10 - l) = -l^2 + 10l.$$

**Step 4 $\frac{1}{2}$ .** **Sketch a graph of the quantity to be maximized or minimized.** This is not absolutely necessary. Sometimes it is impossible. When you can make a rough sketch, this will typically give a very good idea where the maximum or minimum lies. In the example above,  $A(l)$  is a quadratic equation. Because both  $l$  and  $w$  must be nonnegative,  $A(l)$  is only defined on the interval  $0 \leq l \leq 10$ . Thus the graph of  $A(l)$  is a segment of a parabola opening down. The vertex of the parabola is contained in the segment. Thus the vertex is the maximum.

**Step 5. Compute the derivative.** In this case,

$$A'(l) = -2l + 10.$$

**Step 6. Find all critical points, endpoints, discontinuity points, etc.** In most cases, it suffices to find all critical points and endpoints. Occasionally it is also necessary to find all points where  $f'$  is not defined. Rarely it is necessary to also consider discontinuity points (although this is usually so obvious that it does not require a separate step). In this case, the endpoints are  $l = 0$  and  $l = 10$ . The one critical point is  $l = 5$ .

**Step 7. Determine the global maximum or minimum.** Checking all critical points, endpoints, etc., determine the global maximum or the global minimum. In this case,  $A(0)$  equals 0,  $A(10)$  equals 0 and  $A(5)$  equals 25. Thus  $l = 5$  is the global maximum.

**Step 8. Back-substitute.** Plug in the value of the single remaining independent variable to determine the values of the remaining independent variables. In this case,  $w$  equals  $10 - l$ , which is  $10 - 5 = 5$  for  $l = 5$ . Thus, the largest area 25 is enclosed by a square of side length 5.

**Example.** A swimmer is in the water at a distance  $b_1$  meters from shore. She wants to reach a point on land  $b_2$  meters from the water. The point is  $a$  meters parallel to the shore. If the swimmer swims  $v_1$  meters per second and runs  $v_2$  meters per second, at what distance  $x$  from the closest point on shore should she aim to minimize her time to the target? Mathematically, the swimmer is at point  $(0, b_1)$  and wants to reach point  $(a, -b_2)$ , where the shore is the  $x$ -axis. At what point  $(x, 0)$  should she aim?

The constants are  $a$ ,  $b_1$ ,  $b_2$ ,  $v_1$  and  $v_2$ . The variable is  $x$ . It is also convenient to introduce a variable  $d_1$  for the distance from  $(0, b_1)$  to  $(x, 0)$ , and a variable  $d_2$  for the distance from  $(x, 0)$  to  $(a, -b_2)$ . Although not obvious, it is also very convenient to introduce a variable  $\theta_1$  for the acute angle formed by the  $x$ -axis and the line segment joining  $(0, b_1)$  to  $(x, 0)$ . Also introduce  $\theta_2$  for the acute angle formed by the  $x$ -axis and the line segment joining  $(x, 0)$  to  $(a, -b_2)$ .

The time  $T_1$  to swim to point  $(x, 0)$  is,

$$T_1 = \frac{d_1}{v_1} = \frac{1}{v_1}(x^2 + b_1^2)^{1/2}.$$

The time  $T_2$  to run from  $(x, 0)$  to point  $(a, -b_2)$  is,

$$T_2 = \frac{d_2}{v_2} = \frac{1}{v_2}((a - x)^2 + b_2^2)^{1/2}.$$

Thus the total time to reach the target is,

$$T = T_1 + T_2 = \frac{1}{v_1}(x^2 + b_1^2)^{1/2} + \frac{1}{v_2}((a-x)^2 + b_2^2)^{1/2}.$$

The derivative of  $T$  with respect to  $x$  is,

$$\frac{dT}{dx} = \frac{1}{v_1} \left( \frac{1}{2}(x^2 + b_1^2)^{-1/2}(2x) \right) + \frac{1}{v_2} \left( \frac{1}{2}((a-x)^2 + b_2^2)^{-1/2}(-2(a-x)) \right).$$

This simplifies to,

$$\frac{dT}{dx} = \frac{x}{v_1 d_1} - \frac{a-x}{v_2 d_2}.$$

Observe that  $x/d_1$  equals  $\sin(\theta_1)$  and  $(a-x)/d_2$  equals  $\sin(\theta_2)$ . Thus,

$$\frac{dT}{dx} = \frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2}.$$

Technically, there are no endpoints. However, it is obvious that the maximum must occur for  $0 \leq x \leq a$ . Thus these may be taken to be endpoints. The critical value occurs when,

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

This is *Snell's Law* for refraction of light upon crossing from one medium to another. For refraction, a particle of light (perhaps fictitious) replaces the swimmer, a translucent medium of one type replaces the water, and a translucent medium of a second type replaces the land. If light travels with velocity  $v_1$  in the first medium and with velocity  $v_2$  in the second medium, light rays will *refract* upon crossing the boundary between media. Snell's Law describes the angles of this refraction.