

18.01 EXERCISES

Unit 1. Differentiation

1A. Graphing

1A-1 By completing the square, use translation and change of scale to sketch

a) $y = x^2 - 2x - 1$

b) $y = 3x^2 + 6x + 2$

1A-2 Sketch, using translation and change of scale

a) $y = 1 + |x + 2|$

b) $y = \frac{2}{(x-1)^2}$

1A-3 Identify each of the following as even, odd, or neither

a) $\frac{x^3 + 3x}{1 - x^4}$

b) $\sin^2 x$

c) $\frac{\tan x}{1 + x^2}$

d) $(1 + x)^4$

e) $J_0(x^2)$, where $J_0(x)$ is a function you never heard of

1A-4 a) Show that every polynomial is the sum of an even and an odd function.

b) Generalize part (a) to an arbitrary function $f(x)$ by writing

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

Verify this equation, and then show that the two functions on the right are respectively even and odd.

c) How would you write $\frac{1}{x+a}$ as the sum of an even and an odd function?

1A-5. Find the inverse to each of the following, and sketch both $f(x)$ and the inverse function $g(x)$. Restrict the domain if necessary. (Write $y = f(x)$ and solve for y ; then interchange x and y .)

a) $\frac{x-1}{2x+3}$

b) $x^2 + 2x$

1A-6 Express in the form $A \sin(x + c)$

a) $\sin x + \sqrt{3} \cos x$

b) $\sin x - \cos x$

1A-7 Find the period, amplitude, and phase angle, and use these to sketch

a) $3 \sin(2x - \pi)$

b) $-4 \cos(x + \pi/2)$

1A-8 Suppose $f(x)$ is odd and periodic. Show that the graph of $f(x)$ crosses the x -axis infinitely often.

E. 18.01 EXERCISES

1A-9 a) Graph the function f that consist of straight line segments joining the points $(-1, -1)$, $(1, 2)$, $(3, -1)$, and $(5, 2)$. Such a function is called piecewise linear.

b) Extend the graph of f periodically. What is its period?

c) Graph the function $g(x) = 3f((x/2) - 1) - 3$.

~~1B. Velocity and rates of change~~

~~1B-1 A test tube is knocked off a tower at the top of the Green building. (For the purposes of this experiment the tower is 400 feet above the ground, and all the air in the vicinity of the Green building was evacuated, so as to eliminate wind resistance.) The test tube drops $16t^2$ feet in t seconds. Calculate~~

~~a) the average speed in the first two seconds of the fall~~

~~b) the average speed in the last two seconds of the fall~~

~~c) the instantaneous speed at landing~~

~~1B-2 A tennis ball bounces so that its initial speed straight upwards is b feet per second. Its height s in feet at time t seconds is given by $s = bt - 16t^2$~~

~~a) Find the velocity $v = ds/dt$ at time t .~~

~~b) Find the time at which the height of the ball is at its maximum height.~~

~~c) Find the maximum height.~~

~~d) Make a graph of v and directly below it a graph of s as a function of time. Be sure to mark the maximum of s and the beginning and end of the bounce.~~

~~e) Suppose that when the ball bounces a second time it rises to half the height of the first bounce. Make a graph of s and of v of both bounces, labelling the important points. (You will have to decide how long the second bounce lasts and the initial velocity at the start of the bounce.)~~

~~f) If the ball continues to bounce, how long does it take before it stops?~~

~~1C. Slope and derivative~~

~~1C-1 a) Use the difference quotient definition of derivative to calculate the rate of change of the area of a disk with respect to its radius. (Your answer should be the circumference of the disk.)~~

~~b) Use the difference quotient definition of derivative to calculate the rate of change of the volume of a ball with respect to the radius. (Your answer should be the surface area of the ball.)~~

~~1C-2 Let $f(x) = (x - a)g(x)$. Use the definition of the derivative to calculate that $f'(a) = g(a)$, assuming that g is continuous.~~

~~1C-3 Calculate the derivative of each of these functions directly from the definition.~~

~~a) $f(x) = 1/(2x + 1)$~~

~~b) $f(x) = 2x^2 + 5x + 4$~~

~~c) $f(x) = 1/(x^2 + 1)$~~

~~d) $f(x) = 1/\sqrt{x}$~~

~~e) For part (a) and (b) find points where the slope is $+1$, -1 , 0 .~~

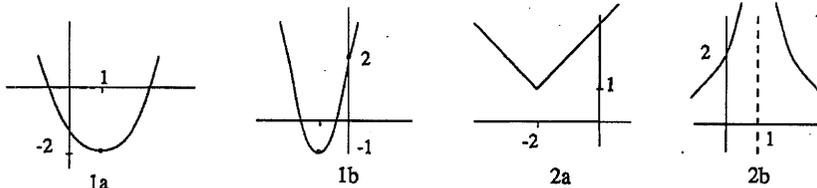
SOLUTIONS TO 18.01 EXERCISES

Unit 1. Differentiation

1A. Graphing

1A-1,2 a) $y = (x - 1)^2 - 2$

b) $y = 3(x^2 + 2x) + 2 = 3(x + 1)^2 - 1$



1A-3 a) $f(-x) = \frac{(-x)^3 - 3x}{1 - (-x)^4} = \frac{-x^3 - 3x}{1 - x^4} = -f(x)$, so it is odd.

b) $(\sin(-x))^2 = (\sin x)^2$, so it is even.

c) $\frac{\text{odd}}{\text{even}}$, so it is odd

d) $(1 - x)^4 \neq \pm(1 + x)^4$: neither.

e) $J_0((-x)^2) = J_0(x^2)$, so it is even.

1A-4 a) $p(x) = p_e(x) + p_o(x)$, where $p_e(x)$ is the sum of the even powers and $p_o(x)$ is the sum of the odd powers

b) $f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$

$F(x) = \frac{f(x) + f(-x)}{2}$ is even and $G(x) = \frac{f(x) - f(-x)}{2}$ is odd because

$$F(-x) = \frac{f(-x) + f(-(-x))}{2} = F(x); \quad G(-x) = \frac{f(-x) - f(-(-x))}{2} = -G(x).$$

c) Use part b:

$$\frac{1}{x+a} + \frac{1}{-x+a} = \frac{2a}{(x+a)(-x+a)} = \frac{2a}{a^2 - x^2} \quad \text{even}$$

$$\frac{1}{x+a} - \frac{1}{-x+a} = \frac{-2x}{(x+a)(-x+a)} = \frac{-2x}{a^2 - x^2} \quad \text{odd}$$

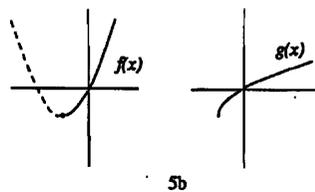
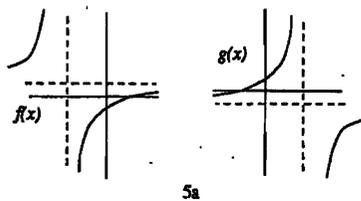
$$\Rightarrow \frac{1}{x+a} = \frac{a}{a^2 - x^2} - \frac{x}{a^2 - x^2}$$

S. 18.01 SOLUTIONS TO EXERCISES

1A-5 a) $y = \frac{x-1}{2x+3}$. Crossmultiply and solve for x , getting $x = \frac{3y+1}{1-2y}$, so the inverse function is $\frac{3x+1}{1-2x}$.

b) $y = x^2 + 2x = (x+1)^2 - 1$

(Restrict domain to $x \leq -1$, so when it's flipped about the diagonal $y = x$, you'll still get the graph of a function.) Solving for x , we get $x = \sqrt{y+1} - 1$, so the inverse function is $y = \sqrt{x+1} - 1$.

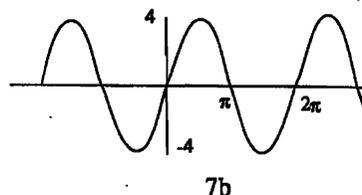
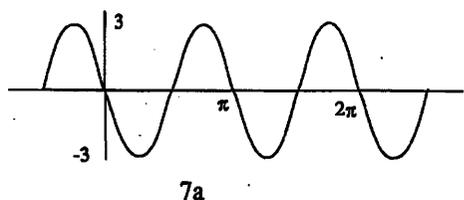


1A-6 a) $A = \sqrt{1+3} = 2$, $\tan c = \frac{\sqrt{3}}{1}$, $c = \frac{\pi}{3}$. So $\sin x + \sqrt{3} \cos x = 2 \sin(x + \frac{\pi}{3})$.

b) $\sqrt{2} \sin(x - \frac{\pi}{4})$

1A-7 a) $3 \sin(2x - \pi) = 3 \sin 2(x - \frac{\pi}{2})$, amplitude 3, period π , phase angle $\pi/2$.

b) $-4 \cos(x + \frac{\pi}{2}) = 4 \sin x$ amplitude 4, period 2π , phase angle 0.

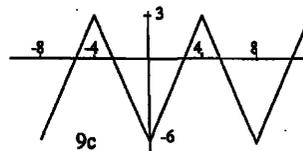
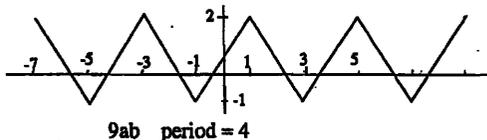


1A-8

$f(x)$ odd $\implies f(0) = -f(0) \implies f(0) = 0$.

So $f(c) = f(2c) = \dots = 0$, also (by periodicity, where c is the period).

1A-9



c) The graph is made up of segments joining $(0, -6)$ to $(4, 3)$ to $(8, -6)$. It repeats in a zigzag with period 8. * This can be derived using:

$x/2 - 1 = -1 \implies x = 0$ and $g(0) = 3f(-1) - 3 = -6$

$x/2 - 1 = 1 \implies x = 4$ and $g(4) = 3f(1) - 3 = 3$

$x/2 - 1 = 3 \implies x = 8$ and $g(8) = 3f(3) - 3 = -6$