

18.01 PRACTICE FINAL, FALL 2003

~~Problem 1~~ Find the following definite integral using integration by parts.

$$\int_0^{\frac{\pi}{2}} x \sin(x) dx.$$

~~Problem 2~~ Find the following antiderivative using integration by parts.

$$\int x \sin^{-1}(x) dx.$$

~~Problem 3~~ Use L'Hospital's rule to compute the following limits.

(a) $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$, $0 < a < b$.

(b) $\lim_{x \rightarrow 1} \frac{4x^3 - 5x + 1}{\ln x}$.

~~Problem 4~~ Determine whether the following improper integral converges or diverges.

$$\int_1^{\infty} e^{-x^2} dx.$$

(Hint: Compare with another function.)

~~Problem 5~~ You wish to design a trash can that consists of a base that is a disk of radius r , cylindrical walls of height h and radius r , and the top consists of a hemispherical dome of radius r (there is no disk between the top of the walls and the bottom of the dome; the dome rests on the top of the walls). The surface area of the can is a fixed constant A . What ratio of h to r will give the maximum volume for the can? You may use the fact that the surface area of a hemisphere of radius r is $2\pi r^2$, and the volume of a hemisphere is $\frac{2}{3}\pi r^3$.

~~Problem 6~~ A point on the unit circle in the xy -plane moves counterclockwise at a fixed rate of $1 \frac{\text{radian}}{\text{second}}$. At the moment when the angle of the point is $\theta = \frac{\pi}{4}$, what is the rate of change of the distance from the particle to the y -axis?

~~Problem 7~~ Compute the following integral using a trigonometric substitution. Don't forget to back-substitute.

$$\int \frac{x^2}{\sqrt{1-x^2}} dx.$$

Hint: Recall the half-angle formulas, $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$, $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$.

~~Problem 8~~ Compute the volume of the solid of revolution obtained by rotating about the x -axis the region in the 1st quadrant of the xy -plane bounded by the axes and the curve $x^4 + r^2 y^2 = r^4$.

~~Problem 9~~ Compute the area of the surface of revolution obtained by rotating about the y -axis the portion of the lemniscate $r^2 = 2a^2 \cos(2\theta)$ in the 1st quadrant, i.e., $0 \leq \theta \leq \frac{\pi}{4}$.

~~Problem 10~~ Compute the area of the *lune* that is the region in the 1st and 3rd quadrants contained inside the circle with polar equation $r = 2a \cos(\theta)$ and outside the circle with polar equation $r = a$.

~~Problem 11~~ Find the equation of every tangent line to the hyperbola C with equation $y^2 - x^2 = 1$, that contains the point $(0, \frac{1}{2})$.

~~(b). As x approaches 1, the numerator approaches $4 - 5 + 1 = 0$, and the denominator approaches $\ln(1) = 0$. The corresponding derivatives are,~~

~~$$\frac{d}{dx}(4x^3 - 5x + 1) = 12x^2 - 5, \quad \frac{d}{dx} \ln(x) = \frac{1}{x}.$$~~

~~Therefore, by L'Hospital's rule,~~

~~$$\lim_{x \rightarrow 1} \frac{4x^3 - 5x + 1}{\ln(x)} = \lim_{x \rightarrow 1} \frac{12x^2 - 5}{\frac{1}{x}} = \frac{12 - 5}{1} = 7.$$~~

Problem 4 ~~Determine whether the following improper integral converges or diverges.~~

~~$$\int_1^{\infty} e^{-x^2} dx.$$~~

~~(Hint: Compare with another function.)~~

Solution ~~Because the integrand is nonnegative, the integral converges if and only if it is bounded. Therefore the comparison test applies. For $x > 1$, $x^2 > x$. Therefore $-x^2 < -x$ and $0 \leq e^{-x^2} < e^{-x}$. Integrating,~~

~~$$\int_1^{\infty} e^{-x} dx = \left(-e^{-x} \Big|_1^{\infty} \right) = (0 + e^{-1}) = e^{-1} < \infty.$$~~

~~Therefore, also $\int_1^{\infty} e^{-x^2} dx$ converges (and is bounded above by e^{-1}).~~

Problem 5 ~~You wish to design a trash can that consists of a base that is a disk of radius r , cylindrical walls of height h and radius r , and the top consists of a hemispherical dome of radius r (there is no disk between the top of the walls and the bottom of the dome; the dome rests on the top of the walls). The surface area of the can is a fixed constant A . What ratio of h to r will give the maximum volume for the can? You may use the fact that the surface area of a hemisphere of radius r is $2\pi r^2$, and the volume of a hemisphere is $\frac{2}{3}\pi r^3$.~~

Solution ~~The area of the base is πr^2 . The area of the sides are $2\pi r h$. The area of the dome is $2\pi r^2$. Therefore we have the equation,~~

~~$$A = \pi r^2 + 2\pi r h + 2\pi r^2 = \pi r(3r + 2h).$$~~

~~It follows that $h = \frac{A}{2\pi r} - \frac{3r}{2}$. The volume of the cylindrical portion of the can is the area of the base times the height, i.e., $\pi r^2 h$. The area of the dome of the can is $\frac{2}{3}\pi r^3$. Therefore the total volume of the can is,~~

~~$$V(r) = \pi r^2 \left(\frac{A}{2\pi r} - \frac{3r}{2} \right) + \frac{2}{3}\pi r^3 = \frac{Ar}{2} - \frac{5\pi r^3}{6}.$$~~

~~The endpoints for r are $r = 0$ and $r = \sqrt{\frac{A}{3\pi}}$. The critical points for r occur when,~~

~~$$\frac{dV}{dr} = \frac{A}{2} - \frac{5\pi r^2}{2} = 0,$$~~

~~i.e., $A = 5\pi r^2$. Since $A = 3\pi r^2 + 2\pi r h$, we conclude that $2\pi r h = A - 3\pi r^2 = 2\pi r^2$. Cancelling, we have that $h = r$. This is contained in the interval for r , moreover geometric reasoning (or the first derivative test) shows this is a maximum for V . Therefore the maximum volume is obtained when $h = r$.~~

Problem 6 ~~A point on the unit circle in the xy -plane moves counterclockwise at a fixed rate of $\frac{1 \text{ radian}}{\text{second}}$. At the moment when the angle of the point is $\theta = \frac{\pi}{4}$, what is the rate of change of the distance from the particle to the y -axis?~~

Solution The coordinates of the point are $(\cos(\theta), \sin(\theta))$. The distance from the y -axis is the absolute value of the x -coordinate. Since the point is in the 1st quadrant, this is just $x = \cos(\theta)$. Therefore the rate of change of the distance is,

$$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = -\sin(\theta) \cdot 1 \frac{\text{radian}}{\text{second}} = -\frac{1}{\sqrt{2}} \frac{\text{radian}}{\text{second}}.$$

~~**Problem 7** Compute the following integral using a trigonometric substitution. Don't forget to back-substitute.~~

$$\int \frac{x^2}{\sqrt{1-x^2}} dx.$$

~~Hint: Recall the half-angle formulas, $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$, $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$.~~

~~**Solution** This integral calls for a trigonometric substitution, $x = \sin(\theta)$, $dx = \cos(\theta)d\theta$. The integral becomes,~~

$$\int \frac{\sin^2(\theta)}{\cos(\theta)} \cos(\theta)d\theta = \int \sin^2(\theta)d\theta.$$

~~By the half-angle formulas, this is,~~

$$\int \frac{1}{2}(1 - \cos(2\theta))d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C.$$

~~Using the double-angle formula, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, and back-substituting $\sin(\theta) = x$ yields,~~

$$\frac{1}{2}\sin^{-1}(x) - \frac{1}{2}x\sqrt{1-x^2} + C.$$

~~**Problem 8** Compute the volume of the solid of revolution obtained by rotating about the x -axis the region in the 1st quadrant of the xy -plane bounded by the axes and the curve $x^4 + r^2y^2 = r^4$.~~

~~**Solution** The curve intersects the y -axis when $r^2y^2 = r^4$, i.e. $y = r$. The curve intersects the x -axis when $x^4 = r^4$, i.e. $x = r$. So the endpoints of the curve are $(0, r)$ and $(r, 0)$. Using the disk method, the volume of the solid is,~~

$$\int_{x=0}^{x=r} \pi y^2 dx.$$

~~Since $y^2 = r^2 - \frac{x^4}{r^2}$, the volume is,~~

$$V = \int_{x=0}^{x=r} \left(\pi r^2 - \frac{\pi x^4}{r^2} \right) dx = \left(\pi r^2 x - \frac{\pi x^5}{5r^2} \right) \Big|_0^r.$$

~~This evaluates to $\pi r^3 - \frac{1}{5}\pi r^3 = \frac{4}{5}\pi r^3$.~~

~~**Problem 9** Compute the area of the surface of revolution obtained by rotating about the y -axis the portion of the lemniscate $r^2 = 2a^2 \cos(2\theta)$ in the 1st quadrant, i.e., $0 \leq \theta \leq \frac{\pi}{4}$.~~

~~**Solution** The polar equation for arclength is $ds^2 = dr^2 + r^2d\theta^2$, which is equivalent to $r^2ds^2 = r^2dr^2 + r^4d\theta^2$. By implicit differentiation,~~

$$2rdr = -4a^2 \sin(2\theta)d\theta, \quad r^2dr^2 = 4a^4 \sin^2(2\theta)d\theta^2.$$

~~Therefore,~~

$$r^2ds^2 = r^2dr^2 + r^4d\theta^2 = 4a^4 \sin^2(2\theta)d\theta^2 + 4a^4 \cos^2(2\theta)d\theta^2 = 4a^4d\theta^2.$$

~~So $ds = \frac{2a^2}{r}d\theta$.~~

~~The area of the surface of revolution is given by,~~

$$\int 2\pi x ds = \int_{\theta=0}^{\theta=\frac{\pi}{4}} 2\pi r \cos(\theta) \frac{2a^2}{r} d\theta = \int_0^{\frac{\pi}{4}} 4\pi a^2 \cos(\theta) d\theta = (4\pi a^2 \sin(\theta)) \Big|_0^{\frac{\pi}{4}}.$$