

# Lesson 24 Roots and Radicals

In Lesson 23 we saw that inverse variation can be expressed as a power function by using negative exponents. We can also use exponents to denote square roots and other radicals.

## $n^{\text{th}}$ Roots

Recall that  $s$  is a square root of  $b$  if  $s^2 = b$ , and  $s$  is a cube root of  $b$  if  $s^3 = b$ . In a similar way, we define the fourth, fifth, or sixth root of a number. For instance, the fourth root of  $b$  is a number  $s$  whose fourth power is  $b$ .

### $N^{\text{th}}$ Roots

The number  $s$  is called an  $n^{\text{th}}$  root of  $b$  if  $s^n = b$ .

We use the symbol  $\sqrt[n]{b}$  to denote the  $n^{\text{th}}$  root of  $b$ . An expression of the form  $\sqrt[n]{b}$  is called a **radical**,  $b$  is called the **radicand**, and  $n$  is called the **index** of the radical.

**Example 1a.**  $\sqrt[4]{81} = 3$  because  $3^4 = 81$

**b.**  $\sqrt[5]{32} = 2$  because  $2^5 = 32$

**c.**  $\sqrt[6]{64} = 2$  because  $2^6 = 64$

**d.**  $\sqrt[4]{1} = 1$  because  $1^4 = 1$

**e.**  $\sqrt[5]{100,000} = 10$  because  $10^5 = 100,000$

**Exercise 1** Evaluate each radical.

**a.**  $\sqrt[4]{16}$

**b.**  $\sqrt[5]{243}$

## Exponential Notation for Radicals

A convenient notation for radicals uses fractional exponents. Consider the expression  $9^{1/2}$ . What meaning can we attach to an exponent that is a fraction? The third law of exponents says that when we raise a power to a power we multiply the exponents together:

$$(x^a)^b = x^{ab}$$

Therefore, if we square the number  $9^{1/2}$ , we get

$$(9^{1/2})^2 = 9^{(1/2)(2)} = 9^1 = 9$$

Thus,  $9^{1/2}$  is a number whose square is 9. But this means that  $9^{1/2}$  is a square root of 9, or

$$9^{1/2} = \sqrt{9} = 3$$

**RQ1a.** Inverse variation can be expressed as a power function by using a \_\_\_\_\_ exponent.

**b.** The number  $n$  in the expression  $\sqrt[n]{b}$  is called the \_\_\_\_\_ of the radical.

**c.** We use \_\_\_\_\_ exponents to denote radicals.

**d.** State the law of exponents that shows that  $a^{1/2}$  is a square root of  $a$ .

## 2 Unit 6 Powers and Roots

In general, any non-negative number raised to the  $1/2$  power is equal to the positive square root of the number, or

$$a^{1/2} = \sqrt{a}$$

■ **Example 2a.**  $25^{1/2} = 5$

**b.**  $-25^{1/2} = -5$

**c.**  $(-25)^{1/2}$  is not a real number

**d.**  $0^{1/2} = 0$  ■

**Exercise 2** Evaluate each power.

**a.**  $4^{1/2}$

**b.**  $4^{-2}$

**c.**  $4^{-1/2}$

**d.**  $(\frac{1}{4})^{1/2}$

The same reasoning works for roots with any index. For instance,  $8^{1/3}$  is the cube root of 8, because

$$(8^{1/3})^3 = 8^{(1/3)(3)} = 8^1 = 8$$

Thus, we make the following definition for fractional exponents.

### Exponential Notation for Radicals

For any natural number  $n \geq 2$ ,

$$a^{1/n} = \sqrt[n]{a}$$

■ **Example 3a.**  $81^{1/4} = \sqrt[4]{81} = 3$

**b.**  $125^{1/3} = \sqrt[3]{125} = 5$  ■

**Caution!** Note that

$$25^{1/2} \neq \frac{1}{2}(25) \quad \text{and} \quad 125^{1/3} \neq \frac{1}{3}(125)$$

An exponent of  $\frac{1}{2}$  denotes the square root of its base, and an exponent of  $\frac{1}{3}$  denotes the cube root of its base. ■

**Exercise 3** Write each power with radical notation, and then evaluate.

**a.**  $32^{1/5}$

**b.**  $625^{1/4}$

Of course, we can use decimal fractions for exponents as well. For example,

$$\sqrt{a} = a^{1/2} = a^{0.5} \quad \text{and} \quad \sqrt[4]{a} = a^{1/4} = a^{0.25}$$

■ **Example 4a.**  $100^{0.5} = \sqrt{100} = 10$

**b.**  $16^{0.25} = \sqrt[4]{16} = 2$  ■

**Exercise 4** Write each power with radical notation, and then evaluate.

**a.**  $100,000^{0.2}$

**b.**

$81^{0.25}$

**RQ2** True or False.

- The exponent  $\frac{1}{n}$  tells us to take a reciprocal.
- The exponent 0.2 denotes a fifth root.
- A power with a negative exponent is not a real number.
- The expressions  $x^{-3}$  and  $x^{1/3}$  are the same.

## Irrational Numbers

What about  $n^{\text{th}}$  roots such as  $\sqrt{23}$  and  $\sqrt[3]{5}$  that cannot be evaluated easily? These are examples of **irrational numbers**. We can use a calculator to obtain decimal approximations for irrational numbers. For example, you can verify that

$$\sqrt{23} \approx 4.796 \quad \text{and} \quad \sqrt[3]{5} \approx 1.710$$

It is not possible to write down an exact decimal equivalent for an irrational number, but we can find an approximation to as many decimal places as we like.

**Caution!** The following keying sequence for evaluating the irrational number  $7^{1/5}$  is *incorrect*.

$$7 \text{ [^] 1 [÷] 5 \text{ [ENTER]}}$$

You can check that this sequence calculates  $\frac{7^1}{5}$ , instead of  $7^{1/5}$ . Recall that, according to the order of operations, powers are computed before multiplications or divisions. Therefore, we must enclose the exponent  $\frac{1}{5}$  in parentheses and enter

$$7 \text{ [^] ([ 1 [÷] 5 ]) \text{ [ENTER]}}$$

Or, because  $\frac{1}{5} = 0.2$ , we can enter

$$7 \text{ [^] 0.2 \text{ [ENTER]}}$$

## Working with Fractional Exponents

Fractional exponents simplify many calculations involving radicals. You should learn to convert easily between exponential and radical notation.

**Example 5** Convert each radical to exponential notation.

a.  $\sqrt[3]{12} = 12^{1/3}$

b.  $\sqrt[4]{2y} = (2y)^{1/4}$  or  $(2y)^{0.25}$

**Caution!** In Example 5b, the parentheses around  $(2y)$  must not be omitted. The expression  $2y^{1/4}$  means  $2\sqrt[4]{y}$ .

Remember that a negative exponent denotes a reciprocal.

**Exercise 5** Convert each radical to exponential notation.

a.  $\frac{1}{\sqrt[5]{ab}}$

b.  $3\sqrt[6]{w}$

## 4 Unit 6 Powers and Roots

**Example 6** Convert each power to radical notation.

a.  $5^{1/2} = \sqrt{5}$

b.  $x^{0.2} = \sqrt[5]{x}$

c.  $2x^{1/3} = 2\sqrt[3]{x}$

d.  $8a^{-1/4} = \frac{8}{\sqrt[4]{a}}$

**Caution!** In Example 6d, the exponent  $-1/4$  applies only to  $a$ , not to

8a. Compare with  $(8a)^{-1/4} = \frac{1}{\sqrt[4]{8a}}$ .

**Exercise 6a.** Convert  $\frac{3}{\sqrt[4]{2x}}$  to exponential notation.

b. Convert  $-5b^{0.125}$  to radical notation.

**RQ3** True or False.

- All expressions of the form  $x^{1/n}$  represent irrational numbers.
- When we enter an exponent  $1/n$  into a calculator, we must enclose it in parentheses.
- In the expression  $16x^{1/4}$ , the exponent does not apply to 16.
- You can use your calculator to find a decimal equivalent for an irrational number.

## Using Fractional Exponents to Solve Equations

We know that raising to powers and taking roots are *inverse operations*, that is, each operation undoes the effects of the other. This relationship is easy to see when we use a fractional exponent for the root. For example, to solve the equation

$$x^5 = 250$$

we would take the fifth root of each side. But instead of using radical notation, we can raise both sides of the equation to the power  $\frac{1}{5}$ . This strategy works because the third law of exponents tells us that  $(x^a)^b = x^{ab}$ , so in particular  $(x^5)^{1/5} = x^{(1/5)(5)} = x^1$ . The solution then looks like this:

$$\begin{aligned}(x^5)^{1/5} &= 250^{1/5} \\ x &\approx 3.017\end{aligned}$$

Thus, to solve an equation involving  $x^n$ , we first isolate the power, then raise both sides to the exponent  $\frac{1}{n}$ .

**Example 7** For astronomers, the mass of a star is its most important property, but also the most difficult to measure directly. Because the numbers involved are so large, astronomers often use solar constants as units of measure: the luminosity of the Sun is 1 *solar luminosity*, and its mass is 1 *solar mass*. With these units, the luminosity of a star is equal to the fourth power of its mass, or  $L = M^4$ . The star Sirius is 23 times brighter than the Sun, so its luminosity is 23 solar luminosities. Estimate the mass of Sirius in units of solar mass.

**Solutions** We substitute the luminosity of Sirius,  $L = 23$ , to get

$$23 = M^4$$

To solve the equation for  $M$ , we raise both sides to the  $\frac{1}{4}$  power.

$$(23)^{1/4} = (M^4)^{1/4}$$

$$2.1899 = M$$

The mass of Sirius is about 2.2 solar masses, or about 2.2 times the mass of the Sun. ■

**RQ4a.** How do we solve the equation  $x^8 = 32$  ?

**b.** Which law of exponents justifies this strategy?

**Exercise 7** The manager of the Atlantis Hotel wants to install in the lobby a spherical fish tank that holds at least 900 cubic feet of water. What should be the radius of the fish tank?

## Power Functions

The size of a diamond is usually given by its weight  $w$  in carats. The diameter,  $D$ , of a diamond cut in a traditional round style is then a power function of its weight, given by

$$D = f(w) = kw^{1/3}$$

■ **Example 8a.** A diamond weighing one-quarter carat has diameter about 4.05 millimeters. Find the constant of proportionality,  $k$ , and write a formula for  $D$  as a function of  $w$ .

**b.** Complete the table with the diameters of diamonds of various weights.

$w$ (carats)	0	0.5	0.75	1.0	1.25	1.5	2.0	2.5	3.0
$D$ (mm)									

**c.** Sketch a graph of the function  $D = f(w)$ .

**d.** What is the diameter of the Hope Diamond, which weighs 45.52 carats and is worth \$300 million?

**Solutions a.** We substitute  $w = 0.25$  and  $D = 4$  into the equation and solve for  $k$ .

$$4.05 = k(0.25)^{1/3}$$

$$k = \frac{4.05}{0.25^{1/3}} = 6.429$$

Rounded to tenths,  $k = 6.4$ , and  $D = 6.4w^{1/3}$ .

**b.** We evaluate the function for each of the weights given in the table. The diameters are rounded to tenths of a millimeter.

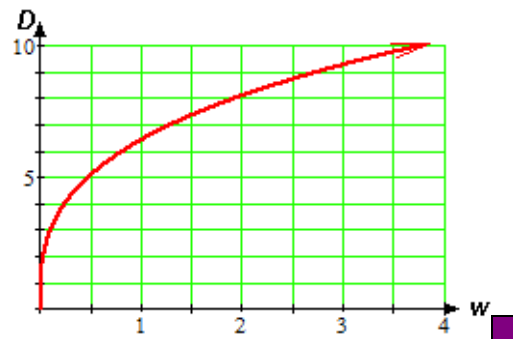
$w$ (carats)	0	0.5	0.75	1.0	1.25	1.5	2.0	2.5	3.0
$D$ (mm)	0	5.1	5.8	6.4	6.9	7.3	8.1	8.7	9.2

**c.** We plot the points from the table and draw a smooth curve to obtain the graph shown.

**d.** We evaluate the function for  $w = 45.52$  to obtain

$$D = 6.4(45.52)^{1/3} = 22.85$$

According to our formula, the Hope Diamond is about 22.85 millimeters in diameter.



## 6 Unit 6 Powers and Roots

Of course, power functions can be expressed using any of the notations we have discussed. The function in Example 8 can be written as

$$f(w) = 6.4w^{1/3} \quad \text{or} \quad f(w) = 6.4w^{0.\overline{33}} \quad \text{or} \quad f(w) = 6.4\sqrt[3]{w}$$

**Exercise 8a.** Complete the table of values for the power function  $f(x) = x^{-1/2}$ .

$x$	0.1	0.25	0.5	1	2	4	8	10	20	100
$f(x)$										

- b. Sketch the graph of  $y = f(x)$ .  
c. Write the formula for  $f(x)$  with a decimal exponent, and with radical notation.

## Solving Radical Equations

A **radical equation** is one in which the variable appears under a square root or other radical. The radical may be denoted by a fractional exponent. For example, the equation

$$5x^{1/3} = 32$$

is a radical equation because  $x^{1/3} = \sqrt[3]{x}$ . To solve the equation, we first isolate the power to get

$$x^{1/3} = 6.4$$

Then we raise both sides of the equation to the reciprocal of  $\frac{1}{3}$ , or 3.

$$\begin{aligned}(x^{1/3})^3 &= 6.4^3 \\ x &= 262.144\end{aligned}$$

**Example 9** When a car brakes suddenly, its speed can be estimated from the length of the skid marks it leaves on the pavement. A formula for the car's speed in miles per hour is  $v = f(d) = (24d)^{1/2}$ , where the length of the skid marks,  $d$ , is given in feet.

- a. If a car leaves skid marks 80 feet long, how fast was the car traveling when the driver applied the brakes?  
b. How far will a car skid if its driver applies the brakes while traveling 80 miles per hour?

**Solutions a.** To find  $v$ , we evaluate the function for  $d = 80$ .

$$\begin{aligned}v &= (24 \cdot 80)^{1/2} && \text{Substitute 80 for } d. \\ &= (1920)^{1/2} && \text{Multiply inside the radical.} \\ &\approx 43.8178046 && \text{Take the square root.}\end{aligned}$$

The car was traveling at approximately 44 miles per hour.

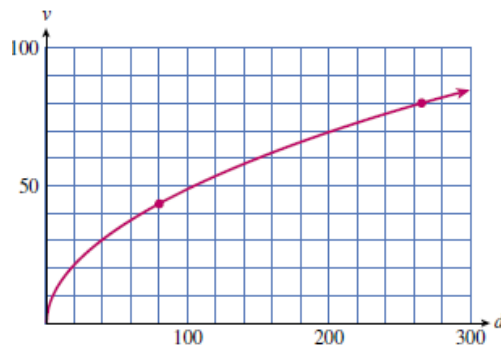
b. We would like to find the value of  $d$  when the value of  $v$  is known. We substitute  $v = 80$  into the formula and solve the equation

$$80 = (24d)^{1/2} \quad \text{Solve for } d.$$

Because  $d$  appears to the power  $\frac{1}{2}$ , we first square both sides of the equation to get

$$\begin{aligned}80^2 &= ((24d)^{1/2})^2 && \text{Square both sides.} \\ 6400 &= 24d && \text{Divide by 24.} \\ 266.\overline{6} &= d\end{aligned}$$

You can check that this value for  $d$  works in the original equation. Thus, the car will skid about 267 feet. A graph of the function  $v = (24d)^{1/2}$  is shown above, along with the points corresponding to the values in parts (a) and (b).



Thus, we can solve an equation where one side is an  $n^{\text{th}}$  root of  $x$  by raising both sides of the equation to the  $n^{\text{th}}$  power. We must be careful when raising both sides of an equation to an even power, as extraneous solutions may be introduced. However, because most applications of power functions deal with positive variables only, they do not usually involve extraneous solutions.

**RQ5a.** How do we solve the equation  $x^{1/6} = 12$ ?

**b.** How do we solve the equation  $\sqrt[6]{x} = 12$ ?

**c.** How do we solve the equation  $x^6 = 12$ ?

**d.** How do we evaluate the function  $f(x) = x^{1/6}$  for  $x = 12$ ?

**Exercise 9** Solve  $5\sqrt[4]{x-1} = 10$

Isolate the radical.

Raise both sides to the fourth power.

Complete the solution.

## Roots of Negative Numbers

You already know that  $\sqrt{-9}$  is not a real number, because there is no real number whose square is  $-9$ . Similarly,  $\sqrt[4]{-16}$  is not a real number, because there is no real number  $r$  for which  $r^4 = -16$ . (Both of these radicals represent complex numbers.) In general, we cannot find an even root (square root, fourth root, and so on) of a negative number.

On the other hand, every positive number has *two* even roots that are real numbers. For example, both 3 and  $-3$  are square roots of 9. The symbol  $\sqrt{9}$  refers only to the positive, or **principal root** of 9. If we want the negative square root of 9 we must write  $-\sqrt{9} = -3$ . Similarly, both 2 and  $-2$  are fourth roots of 16, because  $2^4 = 16$  and  $(-2)^4 = 16$ . The symbol  $\sqrt[4]{16}$  refers to the principal, or positive, fourth root only. Thus,

$$\sqrt[4]{16} = 2 \quad \text{and} \quad -\sqrt[4]{16} = -2$$

Things are simpler for odd roots (cube roots, fifth roots, and so on). Every real number, whether positive, negative, or zero, has exactly one real-valued odd root. For example,

$$\sqrt[5]{32} = 2 \quad \text{and} \quad \sqrt[5]{-32} = -2$$

Here is a summary of our discussion.

## 8 Unit 6 Powers and Roots

### Roots of Real Numbers

1. Every positive number has two real-valued roots, one positive and one negative, if the index is even.
2. A negative number has no real-valued root if the index is even.
3. Every real number, positive or negative, has exactly one real-valued root if the index is odd.

■ **Example 10a.**  $-\sqrt[4]{625} = -5$

**b.**  $\sqrt[4]{-625}$  is not a real number.

**c.**  $\sqrt[5]{-1} = -1$

**d.**  $\sqrt[4]{-1}$  is not a real number. ■

The same principles apply to powers with fractional exponents. Thus

$$(-32)^{1/5} = -2$$

but  $(-64)^{1/6}$  is not a real number. On the other hand,

$$-64^{1/6} = -2$$

because the exponent  $1/6$  applies only to 64, and the negative sign is applied after the root is computed.

**RQ6** Fill in the blanks.

- a. We cannot take an \_\_\_\_\_ root of a \_\_\_\_\_ number.
- b. The principal root is the \_\_\_\_\_ root.
- c. An odd root of a negative number is \_\_\_\_\_.
- d. An  $n^{\text{th}}$  root is the same as the \_\_\_\_\_ power.

**Exercise 10** Evaluate each power, if possible.

**a.**  $-81^{1/4}$

**b.**  $(-81)^{1/4}$

**c.**  $-64^{1/3}$

**d.**  $(-64)^{1/3}$

### Answers to Exercises

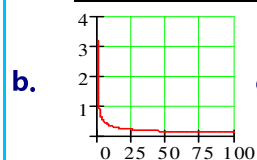
**1a.** 2   **b.** 3   **2a.** 2   **b.**  $\frac{1}{16}$    **c.**  $\frac{1}{2}$    **d.**  $\frac{1}{2}$    **3a.**  $\sqrt[5]{32} = 2$    **b.**  $\sqrt[4]{625} = 5$

**4a.**  $\sqrt[5]{100,000} = 10$    **b.**  $\sqrt[4]{81} = 3$    **5a.**  $(ab)^{-1/5}$    **b.**  $3w^{1/6}$

**6a.**  $3(2x)^{1/4}$    **b.**  $-5\sqrt[8]{b}$    **7.** 6 ft

**8a.**

$x$	0.1	0.25	0.5	1	2	4	8	10	20	100
$f(x)$	3.16	2	1.41	1	0.71	0.5	0.35	0.32	0.22	0.1



**c.**  $f(x) = x^{-0.5}; f(x) = \frac{1}{\sqrt{x}}$

**9.** 17   **10a.** -3

**b.** undefined

**c.** -4

**d.** -4

**Now go to WeBWork and do the Reading Assignment for this Lesson.**

## Homework 24

□ Write each expression as a power function.

1. a.  $g(x) = 3.7 \sqrt[3]{x}$       b.  $H(x) = \sqrt[4]{85x}$       c.  $F(t) = \frac{25}{\sqrt[5]{t}}$

2. Match each function with the description of its graph in the first quadrant.

I.  $f(x) = x^2$     II.  $f(x) = x^{-2}$     III.  $f(x) = x^{1/2}$     IV.  $f(x) = x^{-1/2}$

- a. Increasing and concave up      b. Increasing and concave down  
c. Decreasing and concave up      d. Decreasing and concave down

- A** 3. The period of a pendulum is the time it takes for the pendulum to complete one entire swing, from left to right and back again. The greater the length,  $L$ , of the pendulum, the longer its period,  $T$ . In fact, if  $L$  is measured in feet, then the period is given in seconds by

$$T = 2\pi \sqrt{\frac{L}{32}}$$

- a. Write the formula for  $T$  as a power function in the form  $f(x) = kx^p$ .  
b. The Foucault pendulum in the Convention Center in Portland, Oregon is 90 feet long. What is its period?  
c. Choose a suitable window and graph the function  $T = f(L)$ . Label the point corresponding to part (b) on the graph.
4. The rate,  $r$ , in feet per second, at which water flows from a fire hose varies directly with the square root of the water pressure,  $P$ , in psi (pounds per square inch).

$P$ (psi)	10	20	30	40
$r$ (ft/sec)	38.3	54.1	66.3	76.5

- a. Find the value of  $k$  and write a power function relating the variables.  
b. What is the rate of water flow at a typical water pressure of 60 psi?  
c. Graph your function and verify your answer to part (b) on the graph.
- A** 5. Thanks to improvements in technology, the annual electricity cost of running most major appliances has decreased steadily since 1970. The average annual cost of running a refrigerator is given in dollars by the function

$$C(t) = 148 - 28t^{1/3}$$

where  $t$  is the number of years since 1970.

- a. How much did it cost to run a refrigerator in 1980? In 1990?  
b. When was the cost of running a refrigerator half of the cost in 1970? If the cost continues to decline according to the given function, when will it cost \$50 per year to run a refrigerator?  
c. Graph the function  $C(t)$ . Do you think that the cost will continue to decline indefinitely according to the given function? Why or why not?

## 10 Unit 6 Powers and Roots

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6. A rough estimate for the radius of the nucleus of an atom is provided by the formula

$$r = kA^{1/3}$$

where  $A$  is the mass number of the nucleus, and  $k \approx 1.3 \times 10^{-13}$  centimeter.

- Estimate the radius of the nucleus of an atom of iodine-127, which has mass number 127. If the nucleus is approximately spherical, what is its volume?
- The nuclear mass of iodine-127 is  $2.1 \times 10^{-22}$  gram. What is the density of the nucleus? (Density is mass per unit volume.)
- Complete the table of values for the radii of various radioisotopes.

Element	Carbon	Potassium	Cobalt	Technetium	Radium
Mass number, $A$	14	40	60	99	226
Radius, $r$					

- Sketch a graph of  $r$  as a function of  $A$ . (Use units of  $10^{-13}$  centimeter on the vertical axis.)
7. a. Graph the functions  $f(x) = 4\sqrt[3]{x-9}$  and  $g(x) = 12$  in the window
- $$\begin{array}{ll} X_{\min} = 0 & X_{\max} = 47 \\ Y_{\min} = -8 & Y_{\max} = 16 \end{array}$$
- Use the graph to solve the equation  $4\sqrt[3]{x-9} = 12$ .
  - Solve the equation algebraically.
8. a. Graph the functions  $f(x) = 6 + 2\sqrt[4]{12-x}$  and  $g(x) = 10$  in the window
- $$\begin{array}{ll} X_{\min} = -27 & X_{\max} = 20 \\ Y_{\min} = 4 & Y_{\max} = 12 \end{array}$$
- Use the graph to solve the equation  $6 + 2\sqrt[4]{12-x} = 10$ .
  - Solve the equation algebraically.

**Now go to WeBWork and do the Skills  
Practice for this Homework.**